Who Saves More: The Naive or the Sophisticated Agent?*

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Abstract

We consider an additively time-separable life-cycle model for the family of power period utility functions u such that $u'(c) = c^{-\theta}$ for resistance to inter-temporal substitution of $\theta > 0$. The utility maximization problem over life-time consumption is dynamically inconsistent for almost all specifications of effective discount factors. Pollak (1968) shows that the savings behavior of a sophisticated agent and her naive counterpart is always identical for a logarithmic utility function (i.e., for $\theta = 1$). As an extension of Pollak's result we show that the sophisticated agent saves more (less) in every period than her naive counterpart whenever $\theta > 1$ ($\theta < 1$) irrespective of the specification of discount factors. We further show that this finding extends to an environment with risky returns and dynamically inconsistent Epstein-Zin-Weil preferences.

JEL Classification: D15, D91, E21.

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1 Introduction

How households consume and save over the life-cycle and how time preferences and beliefs about the future affect these decisions are classical economic questions. The workhorse model to address this problem of inter-temporal allocation is the life-cycle model of Modigliani and Brumberg (1954) and Ando and Modigliani (1963). Standard models consider an expected utility maximizing agent with an additively separable per period utility function. The agent's future utility is discounted by a rate of time-preference typically described by an exponential discount function following Samuelson (1937). The more general effective discount function also incorporates the belief to survive into the future together with the pure time discount factor. Following Muth (1961) it has become standard to express survival beliefs as objective (additive) survival probabilities.

This paper extends the standard model by allowing for arbitrary effective discount factors. More precisely, we study an additively time-separable life-cycle model with final period $T \geq 1$ such that every h-old agent's (remaining) life-time utility over the consumption stream $(c_h, c_{h+1}..., c_T) \in \mathbb{R}_{>0}^{T-h+1}$ is given as

$$U_h\left(c_h, c_{h+1}..., c_T\right) = \sum_{t=h}^{T} \rho_{h,t} u\left(c_t\right), \tag{1}$$

whereby the age-dependent effective discount factors must only satisfy $\rho_{h,t} > 0$ and $\rho_{t,t} = 1$. There exists an initial amount of total wealth $w_0 > 0$[1] that the agent can spend over her life-cycle so that the budget constraint becomes

$$w_{t+1} = w_t - c_t \geq 0 \quad \text{for } t \in \{0, 1, \ldots, T-1\}.$$

Effective discount factors capture in deterministic models pure time-discounting and in models with survival uncertainty, typically, a combination of pure time-discounting and survival beliefs.[2] Because the discount factors of the h-old agent, $h = 0, ..., T-1$, can be any strictly positive real-numbers, our model is very general and it encompasses relevant extensions of the standard model such as, e.g., (quasi-)hyperbolic time-discounting models

[1] Total wealth is the sum of financial wealth and discounted future risk-free labor income.

[2] Compare, e.g., Halevy (2008), Eppert et al. (2011), Saito (2011) and Chakraborty et al. (2020) who discuss the delicate relationship between pure time-preferences and preferences under uncertainty or/and risk.

(cf. Phelps and Pollak 1968; Laibson 1997; 1998; O'Donoghue and Rabin 1999; Harris and Laibson 2001) and Choquet expected utility or/and Prospect theory life-cycle models with non-additive subjective survival beliefs (cf. Bleichrodt and Eeckhoudt 2006; Groneck et al. 2016 and references therein). To make this latter point explicit, we show in the Appendix A that (1) represents the preferences of an h-old Choquet expect utility (=CEU) decision maker whose effective discount factors are given as

$$\rho_{h,t} = \beta_{h,t}\nu_{h,t} \tag{2}$$

where $\beta_{h,t}$ stands for pure time-discounting between present age h and future age t while $\nu_{h,t}$ stands for the decision maker's non-additive belief to survive from age h to age t.[3]

For analytical convenience we restrict attention to period-utility functions belonging to the family of iso-elastic power utility functions, that is, $u(c)$ must be differentiable on $\mathbb{R}_{>0}$ such that $u'(c) = c^{-\theta}$ for concavity parameter $\theta > 0$. In static decision situations under risk or/and uncertainty, θ would correspond to the *constant relative risk aversion* (=CRRA) coefficient so that greater values of θ express a greater aversion against risk or/and uncertainty. In the context of intertemporal consumption choices, θ measures the *resistance* to inter-temporal substitution, respectively its inverse $1/\theta$ is the *elasticity of inter-temporal substitution* (=IES). Thus, a lower IES describes a decision maker who is less willing to change her consumption allocation over time. Overall, a greater value of the concavity parameter θ means that the agent is more eager to smooth out consumption over different states of the world as well as over different time periods.

A challenge when solving this general life-cycle model is the fact that it is, generically, *dynamically inconsistent* for $T \geq 2$: The optimal consumption plan from the perspective of some ex ante agent does—for almost all specifications of discount factors—not coincide with the optimal consumption plan from the perspective of some ex post agent. To be precise, in our model *dynamic consistency at age h* is formally equivalent to the following

[3] The crucial structural condition for this derivation is *additive separability* of the decision maker's Bernoulli utility function—which is defined over truncated consumption streams—into per-period utility/felicity functions. Such additive separability is, in general, not satisfied for Epstein-Zin-Weil preferences, which we discuss in Section 5.

system of equations (cf. Proposition 4):

$$\sum_{k=t+1}^{T}\left(\frac{\rho_{h,k}}{\rho_{h,t}}\right)^{\frac{1}{\theta}}=\sum_{k=t+1}^{T}\left(\rho_{t,k}\right)^{\frac{1}{\theta}} \text{ for all } t\geq h+1, \tag{3}$$

which is violated for almost all values of the discount factors whenever $h \leq T-2$.[4] Note that dynamic consistency of the model at all ages is always guaranteed under the following specification of discount factors: for all h

$$\frac{\rho_{h,t+1}}{\rho_{h,t}}=\rho_{t,t+1} \text{ for all } t\geq h+1.$$

Whereas this sufficiency condition for dynamic consistency is satisfied for standard exponential time-discounting combined with additive survival beliefs, it does typically not hold for (quasi-)hyperbolic time-discounting models or for models with non-additive survival beliefs. To deal with the generic case of dynamic inconsistency, we solve the life-cycle model for the realized consumption paths of a sophisticated and a naive agent, respectively. Despite the fact that the sophisticated agent and her naive counterpart share the same preferences over consumption streams, the realized consumption paths of both agent types result from very different optimization problems. The sophisticated agent—who is aware of the deviating incentives of her future selfs—chooses her per-period consumption as if she plays a strategic game against her future selfs. In contrast, the naive agent chooses her per-period consumption under the misperception that her future selfs will stick to the consumption plan that is optimal from her ex ante perspective.

How does the savings behavior of the sophisticated agent compare with that of her naive counterpart? In models with a *presence bias*—induced, e.g., by hyperbolic and quasi-hyperbolic time-discounting—one would intuitively think that sophisticated agents save more than their naive counterparts. More generally, one would probably expect that the answer to the posed question depends on several model parameters like survival beliefs and pure time discount functions. However, this intuition is flawed. A remarkable result by Pollak (1968) already shows that the realized consumption paths of naive and sophisticated agents coincide for a logarithmic period utility function irrespective of survival

[4] On the other hand, the model is always dynamically consistent at age $h \in \{T-1, T\}$, because dynamic inconsistencies can only arise in models with at least three (remaining) periods.

beliefs and time-discounting. That is, although both agent types solve very different life-cycle decision problems, the logarithmic utility function ensures that the actual outcome of realized consumption in every period coincides for the sophisticated and naive agent irrespective of the specification of their shared effective discount factors.

We extend Pollak's (1968) result from the special case of logarithmic utility, i.e., $\theta = 1$, to the whole family of iso-elastic power utility functions with $\theta > 0$. As our main finding we establish that, somewhat surprisingly, the value of the concavity parameter θ completely determines whether the naive or the sophisticated agent saves more in any given period: The sophisticated agent saves in every period more than her naive counterpart if $\theta > 1$; respectively, she saves less if $\theta < 1$. In other words, irrespective of the specification of survival beliefs and pure time-discount factors, the sophisticated agent saves more than her naive counterpart if and only if the period power utility function is more concave than the logarithmic function. To be specific, denote by m_h^s the *marginal propensity to consume* (=MPC) of the h-old sophisticated and by m_h^n the MPC of the h-old naive agent. As our main insight we derive the following theorem.

Theorem 1. *For all (arbitrary) specifications of the effective discount factors we have at every age* $h \leq T - 2$*:*[5]

(i) $\theta < 1$ *implies* $m_h^n \leq m_h^s$*;*

(ii) $\theta > 1$ *implies* $m_h^n \geq m_h^s$*.*

We find it instructive to present two very different proofs of Theorem 1. Proof One is surprisingly basic as it only uses the linearity of the consumption rule in wealth levels for iso-elastic power utility functions to show how an ex ante deviation from the naive agent's consumption rule would affect the sophisticated agent's life-time utility. Proof One establishes that a sophisticated agent has in every period no incentive to choose a strictly greater (smaller) MPC than her naive counterpart whenever $\theta < 1$ ($\theta > 1$). By its very design, Proof One can only establish weak inequalities between the MPCs of the naive and sophisticated agent, respectively.

In contrast, Proof Two is based on a backward induction argument that fully exploits the recursive structure of the agents' consumption problems. It can therefore generate

[5] At ages $h \in \{T-1, T\}$ we always have $m_h^n = m_h^s$ irrespective of the value of θ.

additional insights regarding the question: Under which conditions will a sophisticated agent of a specific age choose a strictly greater (smaller) MPC than her naive counterpart? Based on a lemma derived in Proof Two, we establish the following strict relationship which holds generically on the space of all discount factors: At any age $h \leq T-2$ the h-old sophisticated agent with $h \leq T-2$ will save strictly more (strictly less) than her naive counterpart if and only if $\theta > 1$ ($\theta < 1$).

As a generalization of (1) we consider Epstein-Zin-Weil (=EZW) preferences (Epstein and Zin 1989; Epstein and Zin 1991; Weil 1989) with arbitrary discount-factors such that the h-old agent's utility is recursively defined as

$$U_t^h = u(c_t) + \frac{\rho_{h,t+1}}{\rho_{h,t}} \frac{1}{1-\theta} \left(\mathbb{E}\left((1-\theta) U_{t+1}^h \right)^{\frac{1-\sigma}{1-\theta}} \right)^{\frac{1-\theta}{1-\sigma}} \quad \text{for all } t \geq h \tag{4}$$

where u is an iso-elastic power utility function with concavity parameter $\theta \neq 1$ and the expectation is taken with respect to risky returns.[6] EZW preferences disentangle risk aversion, expressed by the parameter $\sigma > 0$, from resistance against inter-temporal substitution as measured by θ. Remarkably, the findings from Theorem 1 obtained for the additively separable utility function (1) carry exactly over to dynamically inconsistent EZW life-cycle models. That is, the question whether the naive or the sophisticated agent saves more in a dynamically inconsistent EZW life-cycle model is also completely determined by the value of the parameter θ either being smaller or greater than one.

The remainder of our analysis proceeds as follows. Section 2 solves the model for the realized consumption path of the sophisticated agent as well as for the planned versus realized consumption paths of the naive agent. Section 3 formally defines—and discusses—dynamic consistency versus inconsistency of our life-cycle model in terms of the realized versus planned MPCs of the naive agent. Section 4 comprehensively answers our research question: Who saves more: The naive or the sophisticated agent? Section 5 extends our main result to an EZW life-cycle model with arbitrary discount factors. Section 6 concludes. In a decision-theoretic Appendix A we derive the structural expression (2)

[6] Although we restrict attention to a risky endowment process with risky asset returns and an optimal portfolio choice, our analysis of EZW preferences also encompasses models with risky human capital whenever the human capital production function is linear, cf. Krebs (2003).

for effective discount factors under the assumption that CEU decision makers have non-additive survival beliefs as well as additively time-separable preferences over Savage acts whose outcomes are truncated consumption streams. Appendix B contains mathematical proofs.

2 The Life-Cycle Model

2.1 Optimal Consumption Plan

For fixed period consumption c_t and wealth w_t let

$$c_t = m_t w_t$$

where m_t denotes the agent's *marginal propensity to consume* (MPC). Because the optimal period consumption is linear in total wealth for power period utility functions, it will sometimes be convenient to consider MPCs rather than absolute consumption levels.[7] Expressed in terms of MPCs for the periods $h+1, ..., T$ and period h wealth lifetime utility (1) of the h-old agent from consumption stream $(c_h, ..., c_T)$ becomes

$$U_h\left(c_h; m_{h+1}, ..., m_T, w_h\right) = u(c_h) + \sum_{t=h+1}^{T} \rho_{h,t} u\left(\left(w_h - c_h\right) m_t \prod_{j=h+1}^{t-1} \left(1 - m_j\right)\right).$$

Next we derive the MPCs that would maximize this utility function from the perspective of the h-old agent.

For $h = T$, we trivially have as optimal consumption

$$c_T^{*,h}\left(w_T\right) = m_T^{*,h} w_T$$

with optimal MPC $m_T^{*,h} = 1$. For $h < T$, the optimal period h consumption $c_h^{*,h}$ from the perspective of the h-old agent is pinned down by the following FOC:

$$\left.\frac{dU_h\left(c_h; m_{h+1}, ..., m_T, w_h\right)}{dc_h}\right|_{c_h = c_h^{*,h}} = 0$$

$$\Leftrightarrow$$

[7] Linearity of consumption policy functions in models with a deterministic labor income stream and no borrowing constraints is a well-established result in the consumption literature, cf., e.g., Deaton (1992).

$$u'(c_h^{*,h}) = \sum_{t=h+1}^{T} \rho_{h,t} u' \left(\left(w_h - c_h^{*,h} \right) m_t \prod_{j=h+1}^{t-1} (1 - m_j) \right) \left(m_t \prod_{j=h+1}^{t-1} (1 - m_j) \right),$$

which becomes for the power period utility function

$$\left(c_h^{*,h} \right)^{-\theta} = \left(w_h - c_h^{*,h} \right)^{-\theta} \sum_{t=h+1}^{T} \rho_{h,t} \left(m_t \prod_{j=h+1}^{t-1} (1 - m_j) \right)^{1-\theta}.$$

Solving for $c_h^{*,h}$ results in

$$c_h^{*,h} \left(m_{h+1}, ..., m_T, w_h \right) = m_h^{*,h} \left(m_{h+1}, ..., m_T \right) w_h$$

such that the optimal period h MPC for fixed period $h+1, ..., T$ MPCs is given as

$$m_h^{*,h} \left(m_{h+1}, ..., m_T \right) = \frac{1}{1 + \left(\sum_{t=h+1}^{T} \rho_{h,t} \left(m_t \prod_{j=h+1}^{t-1} (1 - m_j) \right)^{1-\theta} \right)^{\frac{1}{\theta}}}.$$

More generally, by the envelope theorem, the optimal period $t \geq h$ consumption from the perspective of the h-old agent given fixed values of $m_{t+1}, ..., m_T$ and wealth w_t is pinned down by

$$\rho_{h,t} \left(c_t^{*,h} \right)^{-\theta} = \left(w_t - c_t^{*,h} \right)^{-\theta} \sum_{s=t+1}^{T} \rho_{h,s} \left(m_t \prod_{j=t+1}^{s-1} (1 - m_j) \right)^{1-\theta}.$$

This gives us the following result.

Proposition 1. *The MPCs $m_t^{*,h}$ that are optimal from the perspective of the h-old agent for fixed $m_{t+1}, ..., m_T$ are given as*

$$m_t^{*,h} \left(m_{t+1}, ..., m_T \right) = \begin{cases} 1 & \text{for } t = T \\ \frac{1}{1 + \left(\sum_{s=t+1}^{T} \frac{\rho_{h,s}}{\rho_{h,t}} \left(m_s \prod_{j=t+1}^{s-1} (1 - m_j) \right)^{1-\theta} \right)^{\frac{1}{\theta}}} & \text{for } h \leq T - 1 \end{cases}$$

For $T \geq 2$ our life-cycle model will be, generically, dynamically inconsistent in the sense that for almost all specifications of discount factors there is some t-old agent with $t > h$, where $h \leq T-2$, who will have a strict incentive to deviate from a consumption plan that would be optimal from the perspective of the h-old agent. To solve for models that might be dynamically inconsistent, the literature distinguishes between the two extreme cases of a naive versus a sophisticated agent (cf. O'Donoghue and Rabin 1999). The remainder of this section defines both types of agents in terms of the optimal MPCs of Proposition 1.

2.2 The Sophisticated Agent

The sophisticated agent correctly anticipates at every age h her future behavior. Denote by m_t^s the realized MPC of the t-old sophisticated agent. Expressed in terms of the optimal MPCs of Lemma 1, the sophisticated agent solves at every age $h \geq 0$ the problem

$$m_h^s = m_h^{*,h}\left(m_{h+1}^s, ..., m_T^s\right),$$

which gives us, by Proposition 1, the following recursive characterization of the realized MPCs of the sophisticated agent.

Proposition 2. *The realized MPCs of the sophisticated agent are given as follows:*

$$m_h^s = \begin{cases} 1 & \text{for } h = T \\ \frac{1}{1+\left(\rho_{h,h+1}\zeta_{h+1}^h\right)^{\frac{1}{\theta}}} & \text{for } h \leq T-1 \end{cases} \tag{5}$$

where ζ_t^h is recursively defined as

$$\zeta_t^h = \begin{cases} 1 & \text{for } t = T \\ {m_t^s}^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}}\left(1 - m_t^s\right)^{1-\theta}\zeta_{t+1}^h & \text{for } t \leq T-1 \end{cases}$$

Solving the model for the sophisticated agent through backward induction is equivalent to solving an extensive form game for the unique subgame-perfect Nash equilibrium where the agents of different ages are different players who can choose MPCs at each information node. The only way how an agent can influence through her chosen MPC the future consumption path in her favor is by restricting the budget, i.e., wealth level, of her future selfs. The MPC m_0^s—being a best response of the 0-old agent against the correctly anticipated MPCs of her future selfs—is therefore a function in m_t^s, for $t \geq h$. On the other hand, the MPCs of future agents do not depend on previously chosen MPCs. This is a consequence of the fact that optimal MPCs are independent of wealth levels for power period utility functions.

2.3 The Naive Agent

Turn now to the naive agent who—possibly incorrectly—assumes that her optimal consumption plan is also optimal from the perspective of all her future selfs. Or put differently, the naive agent completely ignores the possibility that her future selfs might have strict incentives to deviate from her optimal consumption path. Expressed in terms of the optimal MPCs of Proposition 1, the h-old naive agent's *planned* MPCs for $t \geq h$ are characterized as

$$m_t^{n,h} = m_t^{*,h}\left(m_{t+1}^{n,h}, ..., m_T^{n,h}\right). \tag{6}$$

Mathematically equivalently, the h-old naive agent's *planned* MPCs are pinned down by the following FOCs for all t such that $h \leq t < T$:

$$\begin{aligned}
\rho_{h,t}(m_t^{n,h} w_t)^{-\theta} &= \rho_{h,t+1}\left(m_{t+1}^{n,h} w_{t+1}\right)^{-\theta} \\
&\Leftrightarrow \\
\rho_{h,t}(m_t^{n,h} w_t)^{-\theta} &= \rho_{h,t+1}\left(m_{t+1}^{n,h}\left(w_t - m_t^{n,h} w_t\right)\right)^{-\theta} \\
&\Leftrightarrow \\
m_t^{n,h} &= \frac{1}{1 + \left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\right)^{\frac{1}{\theta}}\left(m_{t+1}^{n,h}\right)^{-1}}.
\end{aligned} \tag{7}$$

Substituting

$$m_{t+1}^{n,h} = \frac{1}{1 + \left(\frac{\rho_{h,t+2}}{\rho_{h,t+1}}\right)^{\frac{1}{\theta}}\left(m_{t+2}^{n,h}\right)^{-1}}$$

in (7) gives

$$m_t^{n,h} = \frac{1}{1 + \left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\right)^{\frac{1}{\theta}} + \left(\frac{\rho_{h,t+2}}{\rho_{h,t}}\right)^{\frac{1}{\theta}}\left(m_{t+2}^{n,h}\right)^{-1}}.$$

By repeating this argument until $m_T^{n,h} = 1$, we obtain the following closed form description of planned MPCs

$$m_t^{n,h} = \begin{cases} 1 & \text{for } t = T \\ \frac{1}{1+\sum_{k=t+1}^{T}\left(\frac{\rho_{h,k}}{\rho_{h,t}}\right)^{\frac{1}{\theta}}} & \text{for } t \leq T-1. \end{cases}$$

Let us summarize the above argument, whereby we write $m_h^n = m_h^{n,h}$ for the realized MPCs of the h-old naive agent:

Proposition 3. *The realized MPCs of the naive agent are given as follows:*

(i) *Recursive characterization:*

$$m_h^n = \begin{cases} 1 & \text{for } h = T \\ \frac{1}{1+\left(\sum_{t=h+1}^{T} \rho_{h,t}\left(m_t^{n,h} \prod_{j=h+1}^{t-1}\left(1-m_j^{n,h}\right)\right)^{1-\theta}\right)^{\frac{1}{\theta}}} & \text{for } h \leq T-1 \end{cases}$$

with planned MPCs

$$m_t^{n,h} = \begin{cases} 1 & \text{for } t = T \\ \frac{1}{1+\sum_{k=t+1}^{T}\left(\frac{\rho_{h,k}}{\rho_{h,t}}\right)^{\frac{1}{\theta}}} & \text{for } t \leq T-1 \end{cases}$$

(ii) *Closed form:*

$$m_h^n = \frac{1}{1+\sum_{t=h+1}^{T}\left(\rho_{h,t}\right)^{\frac{1}{\theta}}} \text{ for } h \leq T-1.$$

3 Dynamic Consistency versus Inconsistency

We formally define dynamic consistency versus dynamic inconsistency of the life-cycle model in terms of possible discrepancies between the planned and the realized MPCs of the naive agent. It will be analytical insightful to define these concepts with respect to the agent's age.

Definition 1.

(i) *We say that the model is "dynamically consistent at age h" if and only if*

$$m_t^{n,h} = m_t^n \text{ for all } t \geq h+1.$$

(ii) *Conversely, we say that the model is "dynamically inconsistent at age h" if and only if*

$$m_t^{n,h} \neq m_t^n \text{ for some } t \geq h+1.$$

The model is always dynamically consistent at the ages $h \in \{T, T-1\}$. For $h \leq T-2$ we obtain, by Proposition 3, the following equivalent characterization of dynamic consistency in terms of discount factors.

Proposition 4. *The life-cycle model is dynamically consistent at age* $h \in \{0, ..., T-2\}$ *if and only if, for all* $t \in \{h+1, T-1\}$,

$$\sum_{k=t+1}^{T} \left(\frac{\rho_{h,k}}{\rho_{h,t}}\right)^{\frac{1}{\theta}} = \sum_{k=t+1}^{T} \left(\rho_{t,k}\right)^{\frac{1}{\theta}}. \tag{8}$$

The equations (8) are for every t generically violated over the space of all discount factors so that our model is, for almost all values of discount factors, dynamically inconsistent at any age $h \leq T-2$.

Example 1. To give an illustrative example, let $T \geq 3$ and observe that dynamic consistency at age $h = T-3$ is characterized through the following two equations:

$$m_{T-1}^{n,h} = m_{T-1}^{n} \qquad \Leftrightarrow \qquad \frac{\rho_{h,T}}{\rho_{h,T-1}} = \rho_{T-1,T} \tag{9}$$

and

$$\begin{aligned} m_{T-2}^{n,h} &= m_{T-2}^{n} \\ &\Leftrightarrow \\ \left(\frac{\rho_{h,T-1}}{\rho_{h,T-2}}\right)^{\frac{1}{\theta}} + \left(\frac{\rho_{h,T}}{\rho_{h,T-2}}\right)^{\frac{1}{\theta}} &= \left(\rho_{T-2,T-1}\right)^{\frac{1}{\theta}} + \left(\rho_{T-2,T}\right)^{\frac{1}{\theta}}. \end{aligned} \tag{10}$$

Whenever we find some discount factors that satisfy any of these two both equations, a small perturbation of factors would break down equality. That is, dynamic consistency is non-generic at $h = T-3$ because it breaks down for the perturbed values of discount factors in any open interval—with strictly positive Lebesgue measure—around the original values.□

The standard way to ensure that (8) holds, and thereby dynamic consistency of the model at age h, is to consider discount factors that satisfy[8]

$$\frac{\rho_{h,t+1}}{\rho_{h,t}} = \rho_{t,t+1} \text{ for all } t > h. \tag{11}$$

[8] To see that Condition (11) is only sufficient but not necessary for dynamic consistency at $h \leq T-3$, observe that we have, e.g., in Example 1 $m_{T-2}^{n,h} = m_{T-2}^{n}$ also for discount factors

$$\frac{\rho_{h,T-1}}{\rho_{h,T-2}} = \rho_{T-2,T} \neq \frac{\rho_{h,T}}{\rho_{h,T-2}} = \rho_{T-2,T-1},$$

which violate (11) by construction.

Although the sufficiency condition (11) is non-generic over the space of all discount factors, it is, e.g., satisfied for the standard model which combines exponential time-discounting with (possibly subjective) additive survival beliefs such that

$$\rho_{h,t} = \beta^{t-h} \mu_{h,t}$$

where $\beta \geq 1$ is the pure time-discount factor and $\mu_{h,t}$ is the conditional belief of an h-old agent to survive until age t derived through Bayesian updating from an additive probability measure μ. More generally, we derive in Appendix A—under the assumption that the Bernoulli utility function over consumption streams is additively separable in period utility functions—the following structural form for effective discount factors

$$\rho_{h,t} = \beta_{h,t} \nu_{h,t}$$

for a CEU decision maker with non-additive survival beliefs $\nu_{h,t}$, $t = 1, ..., T$. If these conditional survival beliefs are derived from an application of the *optimistic* Bayesian update rule (Gilboa and Schmeidler 1993) applied to a non-additive probability measure ν, we have that

$$\frac{\nu_{h,t+1}}{\nu_{h,t}} = \nu_{t,t+1} \text{ for all } t > h. \tag{12}$$

Combining the optimistic Bayesian update rule for non-additive beliefs with exponential pure time-discounting gives us effective discount-factors for CEU decision makers that satisfy the sufficiency condition (11) for dynamic consistency of the life-cycle model, i.e.,

$$\frac{\rho_{h,t+1}}{\rho_{h,t}} = \frac{\beta^{t+1-h} \nu_{h,t+1}}{\beta^{t-h} \nu_{h,t}} = \beta \nu_{t,t+1} = \rho_{t,t+1} \text{ for all } t > h.$$

For non-exponential pure time-discounting and/or for non-additive survival beliefs that are not derived from the optimistic update rule, however, the sufficient consistency condition (11) will, in general, fail.[9]

[9] In contrast to the case of an additive measure μ, there exist multiple Bayesian update rules for a non-additive probability measure ν (cf. Appendix A). To be specific, (12) will, e.g., be violated for the *pessimistic* (Gilboa and Schmeidler 1993) and for the *generalized* (Eichberger et al. 2007; 2012) Bayesian update rule. Beyond Bayesian updating, Ludwig and Zimper (2013), Groneck et al. (2016) and Grevenbrock et al. (2020) discuss alternative formations of (non-additive) age-dependent survival beliefs that all violate condition (12).

The realized versus planned MPCs of Proposition 3 illustrate how a dynamic inconsistency of the model might play out for the naive agent. Observe that $m_{h+1}^{n,h} > m_{h+1}^{n}$ for all $h = 0, \ldots, T-2$ if and only if

$$\sum_{k=h+2}^{T} \left(\frac{\rho_{h,k}}{\rho_{h,h+1}} \right)^{\frac{1}{\theta}} < \sum_{k=h+2}^{T} \rho_{h+1,k}^{\frac{1}{\theta}}.$$

If we have, for example, $\frac{\rho_{h,k}}{\rho_{h,h+1}} < \rho_{h+1,k}$ for all $k \geq h+2$, i.e., if discounting exhibits *increasing patience* so that the marginal valuation of saving increases as the agent ages, then the $h+1$-old naive agent will be consuming strictly less than the h-old agent had originally planned for period $h+1$. This model could thus explain the well-known observation that many real-life agents save less than they originally planned, cf. Bernheim (1998), Choi et al. (2006), and Lusardi and Mitchell (2011).

4 Who Saves More: The Naive or the Sophisticated Agent?

We will see that the sophisticated and the naive agent's savings behavior will coincide at all ages if the life-cycle model is dynamically consistent at all ages (cf. Corollary 3 below). Of course, this finding is not surprising. Quite surprising, however, is the following relationship: Even if the life-cycle model is dynamically inconsistent, both types of agents exhibit the same savings behavior whenever the period-utility function is of the logarithmic form. This remarkable finding goes back to the seminal analysis in Pollak (1968).

Theorem 0 (Pollak 1968). *For all (arbitrary) specifications of the effective discount factors we have at every age h:*

$$\theta = 1 \textit{ implies } m_h^n = m_h^s.$$

It is straightforward to verify Pollak's Theorem directly by setting $\theta = 1$ in the MPCs of Propositions 2 and 3 to obtain

$$m_h^s = m_h^n = \begin{cases} 1 & \text{for } h = T \\ \frac{1}{1+\sum_{t=h+1}^{T} \rho_{h,t}} & \text{for } h \leq T-1. \end{cases}$$

For general $\theta \neq 1$ it follows also from the Propositions 2 and 3 that the MPCs of the T- and $T-1$-old agents coincide for the naive and sophisticated type such that

$$\begin{aligned} m_T^n &= m_T^s = 1, \\ m_{T-1}^n &= m_{T-1}^s = \frac{1}{1 + \left(\rho_{T-1,T}\right)^{\frac{1}{\theta}}}. \end{aligned}$$

For any ages $h \leq T-2$, however, it is no longer obvious how the sophisticated and naive savings behavior will compare whenever $\theta \neq 1$. Our next result extends Pollak's Theorem to the whole class of iso-elastic power utility functions, i.e., to all concavity parameter values $\theta \neq 1$.

Theorem 1. *For all (arbitrary) specifications of the effective discount factors we have at every age $h \leq T-2$:*

(i) $\theta < 1$ *implies* $m_h^n \leq m_h^s$*;*

(ii) $\theta > 1$ *implies* $m_h^n \geq m_h^s$*.*

We are going to present two very different proofs of Theorem 1in Appendix B. Our first proof, i.e., "Proof One", exploits the linearity of the sophisticated agent's consumption rule for iso-elastic power utility functions. We ask: Under which conditions on θ will a sophisticated agent never choose a strictly smaller (i.e., strictly greater) MPC than her naive counterpart? By design, the very basic Proof One cannot give us any further insights that go beyond the weak inequalities of Theorem 1.

In contrast, our second proof, i.e., "Proof Two", asks: Under which conditions on θ will a sophisticated agent choose a strictly smaller (i.e., strictly greater) MPC than her naive counterpart? Proof Two uses a backward induction argument which fully exploits the recursive structure of the agents' MPCs as derived in Propositions 2 and 3. Because Proof Two works with a much richer structure than Proof One, it gives us the following additional insights about strict inequalities versus equalities of the MPCs in Theorem 1:

Lemma 1. *Let $h \leq T-2$.*

(i) $\theta < 1$ *implies* $m_h^n < m_h^s$ *if and only if* $m_t^{n,h} \neq m_t^s$ *for some* $t \geq h+1$*.*

(ii) $\theta > 1$ *implies* $m_h^n > m_h^s$ *if and only if* $m_t^{n,h} \neq m_t^s$ *for some* $t \geq h+1$.

(iii) $\theta \neq 1$ *and* $m_h^n = m_h^s$ *if and only if* $m_t^{n,h} = m_t^s$ *for all* $t \geq h+1$.

Let us use the characterizations of Lemma 1 to identify further conditions such that the weak inequalities in Theorem 1 either become strict or hold with equality. At first, observe that $m_{T-1}^s = m_{T-1}^n$ implies

$$m_{T-1}^{n,h} \neq m_{T-1}^s \quad \Leftrightarrow \quad m_{T-1}^{n,h} \neq m_{T-1}^n \quad \Leftrightarrow \quad \frac{\rho_{h,T}}{\rho_{h,T-1}} \neq \rho_{T-1,T}, \tag{13}$$

which gives us by Lemma 1 the following (easy-to-check) sufficiency condition for strict inequalities.

Corollary 1. *Let* $h \leq T-2$. *Whenever the discount factors satisfy inequality (13), we have:*

(i) $\theta < 1$ *implies* $m_h^n < m_h^s$*;*

(ii) $\theta > 1$ *implies* $m_h^n > m_h^s$.

Because inequality (13) holds generically, we can combine these strict inequalities with Theorem 0 by Pollak (1968) to obtain the following statement.

Corollary 2. *Let* $h \leq T-2$. *We have generically that*

$$m_h^n < (>)\, m_h^s \text{ if and only if } \theta < (>)\, 1.$$

Next, recall that we have defined dynamic consistency at age h as

$$m_t^{n,h} = m_t^n \text{ for all } t \geq h+1.$$

Suppose that the model is dynamically consistent at age $h \leq T-2$. By Lemma 1, we have $m_h^n = m_h^s$ if and only if

$$m_t^{n,h} = m_t^n = m_t^s \text{ for all } t \geq h+1. \tag{14}$$

Can we have that the model is dynamically consistent at all ages $t \geq h$ but $m_h^n \neq m_h^s$? The answer is 'no' because $m_h^n \neq m_h^s$ and dynamic consistency at h would imply, by Lemma 1, the existence of some $t \geq h+1$ such that $m_t^{n,h} \neq m_t^s$, a contradiction to (14).

Corollary 3. *Let $h \leq T-2$ and $\theta \neq 1$. If the model is dynamically consistent at all ages $t \geq h$, then $m_h^n = m_h^s$.*

Finally, recall our definition of dynamic inconsistency at age h:

$$m_t^{n,h} \neq m_t^n \text{ for some } t \geq h+1.$$

Suppose that the model is dynamically inconsistent at age $h \leq T-2$. By Lemma 1, we have that $m_h^n = m_h^s$ implies $m_t^n \neq m_t^{n,h} = m_t^s$ for some $t \geq h+1$. Conversely, if the model is dynamically inconsistent at age h, we have that $m_t^n = m_t^s$ for all $t \geq h+1$ implies $m_h^n \neq m_h^s$.

Corollary 4. *If the model is dynamically inconsistent at age $h \leq T-2$, we have:*

(i) *$\theta < 1$ implies $m_t^n < m_t^s$ for some $t \geq h$;*

(ii) *$\theta > 1$ implies $m_t^n > m_t^s$ for some $t \geq h$.*

The remainder of this section presents our two alternative proofs of Theorem 1.

5 Extension to Dynamically Inconsistent Epstein-Zin-Weil Preferences

This section extends our main result to a life-cycle model with random returns and portfolio choice under the assumption that the agent has Epstein-Zin-Weil (EZW) preferences with arbitrary discount factors (Epstein and Zin 1989; Epstein and Zin 1991; Weil 1989). Our extension to EZW life-cycle models builds on two fundamental insights of the seminal work by Samuelson (1969). Namely that, first, with homothetic preferences and i.i.d. returns the portfolio allocation problem can be separated from the inter-temporal consumption-savings problem, and, second, that resulting policy functions for consumption are linear in total wealth. Apart from an additional term which captures the utility consequences of risky returns and the optimal portfolio choice—which is the same for the naive and the sophisticated agent—, the expressions for the marginal propensities to consume out of total wealth of the naive and the sophisticated agent derived from this model are therefore as in our baseline model without risky returns. It is then straightforward to establish that the backward recursive Proof Two readily extends to this setup.

5.1 Epstein-Zin-Weil Preferences with Arbitrary Discount Factors

Building on the axiomatization of dynamically consistent preferences in Kreps and Porteus (1979), Epstein and Zin (1989) and, independently, Weil (1989) have proposed a recursive utility representation that can disentangle risk- from intertemporal attitudes in life-cycle models without any survival risk but with a risky income process in terms of random asset returns. As point of departure, fix the additive probability space $\left(\pi, \hat{\Omega}, \hat{\mathcal{F}}_T\right)$ with information filtration $\left\{\hat{\mathcal{F}}_t\right\}_{t=0,..,T}$, $\hat{\mathcal{F}}_t \subseteq \hat{\mathcal{F}}_{t+1}$, which governs the random asset returns. Let $\mathbb{E}_t$ denote the conditional expectations operator $\mathbb{E}_{\pi\left(\cdot \| \hat{\mathcal{F}}_t\right)}$ where $\pi\left(\cdot \parallel \hat{\mathcal{F}}_t\right)$ becomes for any information $I_t \in \hat{\mathcal{F}}_t$ about period-t asset returns the conditional additive probability measure $\pi\left(\cdot \mid I_t\right)$ updated from π in the standard Bayesian way. Epstein-Zin-Weil (=EZW) preferences belong to a family of utility representations such that the utility U_h^h of an h-old agent is recursively determined by

$$\begin{aligned} f\left(U_t^h\right) &= u\left(c_t\right) + \beta \cdot \phi^{-1}\left(\mathbb{E}_t\left[\phi\left(f\left(U_{t+1}^h\right)\right)\right]\right) \text{ for all } t = h, .., T-1, \qquad (15) \\ f\left(U_T^h\right) &= u\left(c_T\right). \end{aligned}$$

Note that

$$\begin{aligned} f\left(U_t^h\right) &= u\left(c_t\right) + \beta \cdot \phi^{-1}\left(\mathbb{E}_t\left[\phi\left(u\left(c_{t+1}\right) + \beta \cdot \phi^{-1}\left(\mathbb{E}_{t+1}\left[\phi\left(f\left(U_{t+2}^h\right)\right)\right]\right)\right)\right]\right) \qquad (16) \\ &= u\left(c_t\right) + \beta \cdot \phi^{-1}\left(\mathbb{E}_t\left[\phi\left(u\left(c_{t+1}\right) + \beta \cdot \phi^{-1}\left(\mathbb{E}_{t+1}\left[\ldots \beta \cdot \phi^{-1}\left(\mathbb{E}_{T-1}\left[\phi\left(u\left(c_T\right)\right)\right] \ldots\right)\right]\right)\right)\right]\right), \end{aligned}$$

which shows that any strictly increasing $f\left(\cdot\right)$ only impacts on the cardinality of the utility representation (15) so that we can choose an arbitrary strictly increasing function $f\left(\cdot\right)$ and still represent the same preference ordering.

The preferences described by (15) are dynamically consistent because pure time-discounting happens exponential and the additive probability measure governing the return process is updated in a Bayesian fashion. For the special case that $\phi\left(x\right) = x$, (16) becomes

$$f\left(U_t^h\right) = u\left(c_t\right) + \beta \mathbb{E}_t\left[u\left(c_{t+1}\right) + \beta \mathbb{E}_{t+1}\left[\ldots \beta \mathbb{E}_{T-1}\left[u\left(c_T\right)\right] \ldots\right]\right],$$

which is, by the law of iterated expectations for additive probability measures, equivalent to the following additively time-separable utility function

$$f\left(U_t^h\right)=u\left(c_t\right)+\mathbb{E}_t\left[\sum_{s=t+1}^{T}\beta^{s-t}u(c_s)\right]. \tag{17}$$

The literature typically refers to (15) as EZW-preferences whenever

1. the period utility function $u\left(c_t\right)$ belongs to the family of iso-elastic power utility functions such that
$$u\left(c_t\right)=\frac{1}{1-\theta}c_t^{1-\theta}\text{ for }\theta\neq 1,$$
2. the transformative function $\phi\left(\cdot\right)$ is given as[10]
$$\phi\left(x\right)=\frac{1}{1-\sigma}\left(\left(1-\theta\right)x\right)^{\frac{1-\sigma}{1-\theta}}\qquad\Leftrightarrow\qquad\phi^{-1}\left(y\right)=\frac{1}{1-\theta}\left(\left(1-\sigma\right)y\right)^{\frac{1-\theta}{1-\sigma}}, \tag{18}$$
3. the (arbitrary) normalization function $f\left(\cdot\right)$ is chosen as
$$f\left(x\right)=\frac{1}{1-\theta}x^{1-\theta}\Leftrightarrow f^{-1}\left(y\right)=\left(\left(1-\theta\right)y\right)^{\frac{1}{1-\theta}}. \tag{19}$$

Under the above specifications, (15) becomes the familiar definition of EZW preferences put forward in Epstein and Zin (1989, 1991)[11]

$$U_t^h=\left(c_t^{1-\theta}+\beta\left(\mathbb{E}\left[U_{t+1}^{h\,1-\sigma}\right]\right)^{\frac{1-\theta}{1-\sigma}}\right)^{\frac{1}{1-\theta}}\quad\text{for all }t\geq h. \tag{20}$$

Epstein and Zin (1989, 1991) and Weil (1989) show that the parameter $\sigma>0$ is a coefficient of risk-aversion whereas parameter θ is a measure of resistance to inter-temporal substitution. For the parametrization $\sigma=\theta$ the transformative function (18) becomes $\varphi\left(x\right)=x$ so that the additively time-separable utility function (17) is nested as a special case under the EZW preferences (20).

In what follows we deviate in two respects from the standard representation (20) of EZW preferences. Firstly, instead of the normalization (19) we simply choose $f\left(x\right)=x$.

[10] Out of notational simplicity we henceforth only consider $\theta\neq 1,\sigma\neq 1$. The limiting cases $\theta=1,\sigma\neq 1$, $\theta\neq 1,\sigma=1$ and $\theta=\sigma=1$ can be analyzed analogously.

[11] For ease of notation we henceforth drop the time index t in the conditional expectations operator.

This alternative normalization keeps the original EZW preferences (20). Secondly, we generalize the time-discount factor $\beta \in (0,1]$ of the original EZW preferences to arbitrary age-dependent effective discount factors satisfying $\rho_{h,t} > 0$ and $\rho_{t,t} = 1$. As a consequence of these arbitrary discount factors, these generalized EZW preferences generically violate, in contrast to (20), dynamic consistency. Both modifications give us the following model:

Definition 2. *We speak of a homothetic 'EZW life-cycle model with arbitrary discount factors' if the h-old agent's utility* U_h^h *is recursively defined as follows:*

$$U_t^h = \frac{1}{1-\theta} c_t^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \frac{1}{1-\theta} \left(\mathbb{E}\left[\left((1-\theta)\, U_{t+1}^h \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \quad \text{for all } t \geq h. \tag{21}$$

To see that our original life-cycle model with arbitrary discount factors (1) is nested as the special case $\sigma = \theta$, rewrite (21) to obtain

$$\begin{aligned} U_t^h &= \frac{1}{1-\theta} c_t^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \frac{1}{1-\theta} \mathbb{E}\left[(1-\theta)\, U_{t+1}^h \right] \qquad (22) \\ &= u(c_t) + \mathbb{E}\left[\sum_{s=t+1}^{T} \frac{\rho_{h,s}}{\rho_{h,t}} u(c_s) \right]. \end{aligned}$$

For $h = t$ (22) becomes our additively time-separable life cycle model (1) plus the possibility of a random return process governed by the additive probability measure π.

Remark 1. *Note that we derive the specific structural interpretation (2) in terms of survival beliefs and pure time-discount factors only for the effective discount factors in the additively time-separable life-cycle model (1) but not for the effective discount factors in the recursive EZW life-cycle model (21). As it is, there exists an ongoing discussion in the literature regarding interpretational issues of EZW life-cycle models with survival risks rather than with pure time-discounting only (cf. Hugonnier et al. 2013; Córdoba and Ripoll 2017; Bommier et al. 2020; Bommier et al. 2021).*[12] *Since this discussion is beyond the scope of the present paper, the remainder of our analysis simply takes the mathematical form (21) of the EZW life-cycle model with arbitrary discount factors as given whereby we refrain from any deeper structural interpretation of these discount factors.*

[12] In a nutshell, this discussion concerns the question whether homothetic EZW preferences that explicitly incorporate the utility of possible death can be consistent with the natural assumption that 'life is better than death' for parameter values $\sigma \neq \theta$, $\sigma \geq 1, \theta \geq 1$.

5.2 Random Return Process with Portfolio Choice

Let R_t be an i.i.d. risky return factor governed by the additive probability measure $\pi(\cdot) = \pi(\cdot \mid I_t)$ for all $I_t \in \hat{\mathcal{F}}_t$ and let R^f be a risk-free return factor such that $R^f < \mathbb{E}[R_t] = \int R_t d\pi$. The household chooses in period t to invest share α_t in stocks with next period risky return R_{t+1} and $1 - \alpha_t$ in bonds with risk-free return R^f. The stochastic portfolio return on the beginning of period t financial wealth holdings is accordingly $R_t^p = R^f + \alpha_{t-1}\left(R_t - R^f\right)$. Also, let y_t be a possibly time varying deterministic endowment income stream of the agent. The budget constraint in terms of financial wealth a_t is then

$$a_{t+1} = a_t R_t^p(\alpha_{t-1}) + y_t - c_t$$

for $a_0 = 0$ given. In terms of cash on hand $x_t = a_t R_t^p(\alpha_{t-1}) + y_t$ we can rewrite the budget constraint as

$$x_{t+1} = (x_t - c_t)\, R_{t+1}^p(\alpha_t) + y_{t+1}. \tag{23}$$

Since human capital as the discounted sum of future deterministic labor income obeys

$$h_{t+1} = h_t R^f - y_{t+1} \tag{24}$$

we can consolidate budget constraints (23) and (24) to get a budget constraint in terms of total wealth as the sum of cash-on-hand and human capital wealth, $w_t = x_t + h_t$, as

$$w_{t+1} = (w_t - c_t)\, R_{t+1}^p(\hat{\alpha}_t) \tag{25}$$

where $\hat{\alpha}_t = \alpha_t \frac{x_t - c_t}{w_t - c_t}$ is the amount invested in stocks as a fraction of to total savings $w_t - c_t$.

Remark 2. *Our results also hold in a nested model variant without labor income and risky returns (with or without a portfolio choice), where households decumulate a given initial financial wealth endowment over the life-cycle.*

Furthermore, an alternative model giving rise to the same mathematical properties is one with risky labor income generated by risky returns to human capital h_t and a linear human capital production function taking monetary human capital investments i_t as inputs, cf. Krebs (2003). The resource constraint in terms of total wealth is then

$$w_{t+1} = (w_t - c_t)\, R_{t+1}^p(\hat{\alpha}_t, \hat{\omega}_t), \tag{26}$$

where $w_t = (a_t + h_t)\, R_t^p(\hat{\alpha}_{t-1}, \hat{\omega}_{t-1})$, $\hat{\alpha} = \frac{\alpha_t a_t}{a_t + h_t}$, and $\hat{\omega}_t = \frac{h_t}{a_t + h_t}$.[13] *By the homotheticity of preferences the consumption savings and the portfolio allocation problems can be separated (Samuelson, 1969). Although the remainder of our analysis only considers the model with risk-free labor income and associated budget constraint (25), our results extend directly to the model with risky human capital and associated budget constraint (26).*

5.3 Solution

The marginal propensities to consume and a characterization of the optimal portfolio choice resulting from the solution of the consumption savings and portfolio allocation problem of the naive and the sophisticated agents are given in the next proposition, which we formally prove in Appendix B:

Proposition 5. *Consider the EZW life-cycle model with arbitrary discount factors. The marginal propensities to consume are given as follows:*

[13] To derive this, write the financial wealth budget constraint as

$$a_{t+1} = a_t \left(R^f + \alpha_{t-1}\left(R_t - R^f\right)\right) + h_t r_t^h - c_t - i_t,$$

where α_{t-1} is the beginning of period financial wealth share held in stocks as above, and where i_t are the monetary human capital investment expenses. Assume a linear human capital production function

$$h_{t+1} = h_t(1 + \eta) + i_t,$$

where $\eta \sim \Upsilon(\eta)$ is an i.i.d. stochastic shock to human capital accumulation. Adding up the two resource constraints and defining cash-on-hand by $x_t = a_t + h_t$ we obtain

$$x_{t+1} = a_t \left(R^f + \alpha_{t-1}\left(R_t - R^f\right)\right) + h_t \left(1 + r_t^h + \eta\right) - c_t = x_t R_t^p(\hat{\alpha}_{t-1}, \hat{\omega}_{t-1}) - c_t,$$

where $\hat{\alpha}_{t-1} = \frac{\alpha_t a_t}{x_t}$ and $\hat{\omega}_{t-1} = \frac{h_t}{x_t}$ and the portfolio return is thus

$$R_t^p(\hat{\alpha}_{t-1}, \hat{\omega}_{t-1}) = R^f + \hat{\alpha}_{t-1}\left(R_t - R^f\right) + \hat{\omega}_{t-1}\left(R_t^h - R^f\right)$$

for $R_t^h = 1 + r_t^h + \eta$. Finally, let wealth be cash-on-hand cum interest, i.e., $w_t = x_t R_t^p(\hat{\alpha}_{t-1}, \hat{\omega}_{t-1})$, to obtain

$$w_{t+1} = (w_t - c_t)\, R_{t+1}^p(\hat{\alpha}_{t-1}, \hat{\omega}_{t-1}).$$

- *for the sophisticated agent:*

$$m_h^{s,h} = \begin{cases} 1 & \text{for } h = T \\ \frac{1}{1+\left(\rho_{h,h+1}\zeta_{h+1}^h \Theta\left(\hat{\alpha}^*, R^f, R_{h+1}, \pi\right)\right)^{\frac{1}{\theta}}} & \text{for } h < T, \end{cases} \tag{27}$$

where ζ_{h+1}^h follows from the backward recursion in $t = T-1, \ldots, h$

$$\zeta_t^h = m_t^{s^{1-\theta}} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \left(1 - m_t^s\right)^{1-\theta} \zeta_{t+1}^h \cdot \Theta\left(\hat{\alpha}^*, R^f, R_{t+1}, \pi\right) \tag{28}$$

for $\zeta_T^h = 1$, where for all $t = h, \ldots, T-1$

$$\Theta\left(\hat{\alpha}_t^*, R^f, R_{t+1}, \pi\right) = \max_{\hat{\alpha}_t} \left\{ \left(\int \left(R_{t+1}^p(\hat{\alpha}_t)^{1-\theta}\right)^{\frac{1-\sigma}{1-\theta}} d\pi \right)^{\frac{1-\theta}{1-\sigma}} \right\}. \tag{29}$$

- *for the naive agent:*

$$m_t^{n,h} = \begin{cases} 1 & \text{for } t = T \\ \frac{1}{1+\left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\Theta\left(\hat{\alpha}^*, R^f, R_{t+1}, \pi\right)\right)^{\frac{1}{\theta}} m_{t+1}^{n,h^{-1}}} & \text{for } t < T, \end{cases} \tag{30}$$

where $\Theta(\cdot)$ is given by (29).

- *for both agents the optimal portfolio choice $\hat{\alpha}_t^{s^*} = \hat{\alpha}_t^{n^*} = \hat{\alpha}^*$ is the solution to*

$$\int R_{t+1}^p(\hat{\alpha}_t)^{-\sigma} d\pi = 0 \tag{31}$$

We thus find that the separation between risk attitudes as measured by σ and inter-temporal attitudes as measured by θ inherent to EZW preferences is reflected in the solution of this model to the effect that both households choose the same optimal portfolio share $\hat{\alpha}$ as the solution to (31)—which due to the convexity of the function $R_{t+1}^p(\hat{\alpha}_t)^{-\sigma}$ in the portfolio share is decreasing in risk aversion σ—, whereas the relationship between the marginal propensities to consume out of total wealth across the two types of households is exclusively driven by inter-temporal attitudes as measured by θ. Specifically, as in our recursive proof in Subsection B.2 we likewise find that

$$m_h^n \leq m^s \qquad \Leftrightarrow \qquad m_{h+1}^{n,h^\theta} \zeta_{h+1}^h \leq 1$$

and since

$$\frac{\rho_{h,t+1}}{\rho_{h,t}}\Theta\left(\hat{\alpha}^{*},R^{f},R_{t+1},\pi\right)=\left(\frac{1-m_{t}^{n,h}}{m_{t}^{n,h}}\right)^{\theta}m_{t+1}^{n,h}$$

we can use the above in equation (28) to obtain (48). An application of the analogous steps as in the backward recursive proof of Theorem 1 finally gives us the following result:

Corollary 5. *Lemma 1 and thus Theorem 1 extend to the dynamically inconsistent EZW life-cycle model with arbitrary discount factors.*

6 Concluding Remarks

Pollak (1968) shows that—irrespective of the specification of discount factors—the sophisticated agent and her naive counterpart exhibit the same savings behavior whenever their period utility function is logarithmic. We extend Pollak's finding to the class of all iso-elastic power utility functions by showing that the sophisticated agent saves in every period more (less) than her naive counterpart if and only if the resistance to inter-temporal substitution is larger (smaller) than one. We further show that our main result generalizes to models with recursive EZW preferences and risky portfolio returns, which also encompasses models with risky human capital. This confirms the interpretation of our main result in terms of the resistance to inter-temporal substitution. Remarkably, these weak inequalities in savings behavior hold irrespective of the specification of discount factors. The discount factors determine whether the weak inequalities either hold with equality—in the non-generic case of dynamic consistency—or with strict inequality—in the generic case of dynamic inconsistency.

Appendix

A Choquet Expected Utility Preferences

A.1 Non-additive Survival Beliefs

Consider an agent of age $h \geq 0$ and fix some maximal $T \geq 2$ with the interpretation that the agent cannot survive beyond age T. For all ages h we construct the probability spaces $\left(\Omega, \mathcal{F}, \nu^h\right)$ for a non-additive probability measure ν^h which describes the h-old agent's survival beliefs.[14] The state space is given as $\Omega = \{\omega_0, \ldots, \omega_T\}$ and the σ-algebra $\mathcal{F}$ is given as the powerset of Ω. We interpret $D_t = \{\omega_t\}$ as the event in $\mathcal{F}$ that the agent dies at the end of age t. Observe that

$$D_t \cup \cdots \cup D_T \tag{32}$$

stands for the event in $\mathcal{F}$ that the agent of age $h < t$ survives until (at least) the beginning of age t. As a notational convention, we write for the h-old agent's belief to survive until (at least) the beginning of age $t > h$

$$\nu_{h,t} = \nu^h \left(D_t \cup \cdots \cup D_T\right).$$

Definition 3. We consider a system of age-dependent non-additive probability measures $\left\{\nu^h\right\}_{h=1,\ldots,T}$ such that, for every h, $\nu^h : \mathcal{F} \rightarrow [0,1]$ satisfies the following conditions:

(i) Normalization: $\nu_{h,t} = 0$ for all $t < h$, and $\nu_{h,h} = 1$;

(ii) Monotonicity: $\nu_{h,t} \geq \nu_{h,k}$ for $k > t \geq h$;

(iii) Non-degeneracy: $\nu_{h,t} > 0$ for all $t > h$.

The above notion of survival beliefs is very general. It encompasses, for example, survival beliefs derived from a fixed probability weighting function applied to conditional additive probabilities—as in the rank-dependent utility life-cycle models in Bleichrodt

[14] To be precise: when we speak of *non-additive probability measures* we actually mean *not necessarily additive probability measures* as we also allow for the possibility of additive probability measures.

and Eeckhoudt (2006) and in Drouhin (2015)—as well as the calibrated survival beliefs in Ludwig and Zimper (2013) and in Groneck et al. (2016) that are derived from a Choquet Bayesian learning model.

A.2 The Choquet Bayesian Decision Maker

Out of additional consistency considerations it is common practice in the literature to consider a Bayesian decision maker, which imposes a stronger condition on survival beliefs than the above properties (i)-(iii). A Bayesian decision maker is characterized through a Bayesian update rule which generates from a prior belief conditional beliefs (i.e., posteriors) in the light of new information. The information filtration for our life-cycle model is simple: in each period the decision maker 'learns' whether she has survived or not whereby we are only interested in the updated beliefs of the surviving decision maker. That is, the relevant information in any given period t is simply the survival event

$$D_h \cup \cdots \cup D_T$$

according to which the decision maker is h-old. Moreover, the only events that our decision maker cares about are her future survival events (32) for $t > h$. If the prior is some additive probability measure, denoted μ, then there exists a unique Bayesian update rule (i.e., a unique definition of conditional beliefs) according to which

$$\begin{aligned} \mu_{h,t} &= \frac{\mu\left(\left(D_t \cup \cdots \cup D_T\right) \cap \left(D_h \cup \cdots \cup D_T\right)\right)}{\mu\left(D_h \cup \cdots \cup D_T\right)} \\ &= \frac{\mu\left(D_t \cup \cdots \cup D_T\right)}{\mu\left(D_h \cup \cdots \cup D_T\right)}, \end{aligned}$$

implying

$$\mu_{t,t+1} = \frac{\mu_{h,t+1}}{\mu_{h,t}}. \tag{33}$$

In contrast to this unique definition of Bayesian updating for additive probabilities, there exists a multitude of alternative Bayesian update rules for CEU decision makers with non-additive beliefs. Let us briefly explain why.[15] In order to explain Ellsberg paradoxes, CEU preferences must allow for the possibility that Savage's sure-thing principle is

[15] For more details see, e.g., Ghirardato (2002).

violated. Denote by $f_A h$ a Savage act (i.e., a mapping from the state space into the set of consequences) that gives the consequences of act f in the event A and the consequences of act h in the complement event $\Omega \backslash A$. The sure thing principle states that, for all acts f, g, h, h',

$$f_A h \succeq g_A h \Leftrightarrow f_A h' \succeq g_A h'.$$

A Bayesian decision maker is characterized by some rule that determines how her ex ante preferences $\succeq$ are updated to her ex post preferences $\succeq_A$ which are conditional on having observed the event A. If the sure-principle holds, we can unambiguously define, for any h,

$$f_A h \succeq g_A h \Rightarrow f \succeq_A g$$

as unique update rule. In violating the sure-thing principle, however, a CEU decision maker might have the ex ante preferences

$$f_A h \succeq g_A h \text{ and } g_A h' \succeq f_A h'.$$

Under the h-rule preferences would be updated to

$$f_A h \succeq g_A h \Rightarrow f \succeq_A g$$

whereas we would obtain under the h'-rule the opposite ex post preferences

$$g_A h' \succeq f_A h' \Rightarrow g \succeq_A f.$$

In other words, a Bayesian update rule for a CEU decision maker has to specify some act h^*, possibly depending on f, g and A, whose consequences the decision maker associates with the outcomes in the now impossible complement event $\Omega \backslash A$. The fact that we can choose any such h^* explains the multitude of possible update rules for CEU decision makers.

Gilboa and Schmeidler (1993) consider a family of Bayesian update rules for CEU decision makers such that h^* is the same for all f, g and A. Two extreme rules out of this family come with straightforward psychological interpretations. According to the *optimistic* update rule, the act h^* would always result in the worst possible consequences so that the decision maker feels relieved to observe event A instead of the complement event $\Omega \backslash A$. Conversely, the *pessimistic* update rule associates h^* with the best possible consequences so that the decision maker will be disappointed upon observing A.

Denote by

$$\nu^{Bayes}\left(D_t \cup \cdots \cup D_T \mid D_h \cup \cdots \cup D_T\right)$$

the conditional belief of the h-old decision maker to survive until (at least) the beginning of age t such that this belief is formed in accordance with some update rule '*Bayes*'. Fix some update rule *Bayes*. We speak of a Bayesian decision maker if her system of age-dependent beliefs $\left\{\nu^h\right\}_{h=1,\ldots,T}$ satisfies, for all $t > h$,

$$\nu_{h,t} = \nu^{Bayes}\left(D_t \cup \cdots \cup D_T \mid D_h \cup \cdots \cup D_T\right).$$

Next we apply Gilboa and Schmeidler's (1993) formal characterizations of the optimistic and pessimistic update rule, respectively, to survival beliefs.

Optimistic versus pessimistic Bayesian updating of survival beliefs.

(i) *Optimistic update rule:*

$$\nu_{h,t} = \frac{\nu\left(D_t \cup \cdots \cup D_T\right)}{\nu\left(D_h \cup \cdots \cup D_T\right)}.$$

(ii) *Pessimistic update rule:*

$$\nu_{h,t} = \frac{\nu\left(D_0 \cup \cdots \cup D_{h-1} \cup D_t \cup \cdots \cup D_T\right) - \nu\left(D_0 \cup \cdots \cup D_{h-1}\right)}{1 - \nu\left(D_0 \cup \cdots \cup D_{h-1}\right)}. \tag{34}$$

Observe that, analogously to the additive case (33), the optimistic update rule implies

$$\nu_{t,t+1} = \frac{\nu_{h,t+1}}{\nu_{h,t}} \tag{35}$$

for all $t \geq h$.

A.3 The Choquet Expected Utility Life-Cycle Model

Denote by

$$\mathbf{c} = \left(c_h, c_{h+1}..., c_T\right) \tag{36}$$

a consumption plan such that $c_k \geq \eta > 0$, for all $k \in \{h, \ldots, T\}$ whereby the lower bound η is chosen to be non-binding in an optimum. An agent who consumes in accordance with

(36) and dies at the end of age $t \geq h$ obtains the truncated consumption stream $\mathbf{c}^t = (c_h, \ldots, c_t)$ as consequence. Defining the set of consequences X as the set of all truncated consumption streams allows us to interpret a consumption plan (36) as a mapping from the relevant state space into the set of consequences, i.e., $\mathbf{c} : \Omega \backslash \{\omega_0, ..., \omega_{h-1}\} \to X$ such that

	ω_h	ω_{h+1}	$\cdots$	ω_T
$\mathbf{c} = (c_h, c_1..., c_T)$	$\mathbf{c}^h = (c_h)$	$\mathbf{c}^{h+1} = (c_h, c_{h+1})$	$\cdots$	$\mathbf{c}^T = (c_h, c_{h+1}..., c_T)$

That is, we interpret consumption plans as Savage (1954) acts whose deterministic consequences are truncated consumption streams. The states $\{\omega_0, ..., \omega_{h-1}\}$ are irrelevant to the utility of the h-old agent as they have become impossible. We assume that the decision maker prefers to live (i.e., to consume) longer, that is, we assume the following preference ranking over consequences for any h-old agent:

$$\mathbf{c}^T \succeq \cdots \succeq \mathbf{c}^h.$$

Denote by $\{\Omega_0, .., \Omega_m\} \subseteq \mathcal{F}$ a finite partition of the state space Ω such that we have for a measurable real-valued function f

$$f(\Omega_0) \geq \cdots \geq f(\Omega_m).$$

The *Choquet integral* of f with respect to the non-additive probability measure ν^h on $(\Omega, \mathcal{F})$ becomes (Schmeidler 1986):

$$\int f d\nu^h = \sum_{j=0}^{m} f(\Omega_j) \left[\nu^h (\Omega_0, .., \Omega_j) - \nu^h (\Omega_0, .., \Omega_{j-1}) \right]$$

where $\nu^h (\Omega_0, \Omega_{-1}) = 0$. Letting f be a Bernoulli utility function defined over truncated consumption streams results in the following definition of Choquet expected utility (Schmeidler 1989) over consumption plans.

Definition 4. *The Choquet expected utility (CEU) from the consumption plan $c = (c_h, c_{h+1}..., c_T)$ of an h-old agent is given as*

$$CEU_h(\mathbf{c}) = \sum_{j=0}^{T-h} w^h \left(\mathbf{c}^{T-j} \right) \left[\nu_{h,T-j} - \nu_{h,T-j+1} \right] \tag{37}$$

where w^h is a Bernoulli utility function over truncated consumption streams satisfying

$$w^h\left(\mathbf{c}^T\right) \geq \cdots \geq w^h\left(\mathbf{c}^0\right). \tag{38}$$

We follow the literature (cf. Epper et al. 2011; Andreoni and Sprenger 2012) and distinguish between a pure time-discount factor and the agent's survival belief. Denote by $\beta_{h,k} \in (0,1]$, $k = h, ..., T$, the pure time-discount factors of an h-old agent such that $\beta_{h,h} = 1$ and $\beta_{h,k} \geq \beta_{h,k+1}$.

Assumption 1. *The Bernoulli utility of a truncated consumption stream c^{h+t} is additively separable with pure time-discount factors, i.e.,*

$$w^h\left(\mathbf{c}^{h+t}\right) = \sum_{k=h}^{h+t} \beta_{h,k} u\left(c_k\right)$$

for a strictly increasing period-utility function $u : [\eta, \infty) \to \mathbb{R}_{\geq 0}$ for some sufficiently small $\eta > 0$.[16]

By Assumption 1, we can transform the CEU from a consumption plan as follows

$$\begin{aligned}
&CEU_h\left(\mathbf{c}\right)\\
&= \sum_{j=0}^{T} w^h\left(\mathbf{c}^{h+T-j}\right)\left[\nu_{h,T-j} - \nu_{h,T-j+1}\right]\\
&= \left(\sum_{k=h}^{T} \beta_{h,k} u\left(c_k\right)\right)\nu_{h,T} + \left(\sum_{k=h}^{T-1} \beta_{h,k} u\left(c_k\right)\right)\left[\nu_{h,T-1} - \nu_{h,T}\right] + \cdots\\
&= \left[\left(\sum_{k=h}^{T} \beta_{h,k} u\left(c_k\right)\right) - \left(\sum_{k=h}^{T-1} \beta_{h,k} u\left(c_k\right)\right)\right]\nu_{h,T} + \left[\left(\sum_{k=h}^{T-1} \beta_{h,k} u\left(c_k\right)\right) - \left(\sum_{k=h}^{T-2} \beta_{h,k} u\left(c_k\right)\right)\right]\nu_{h,T-1} + \cdots\\
&= \sum_{t=h}^{T} \beta_{h,t}\nu_{h,t} u\left(c_t\right).
\end{aligned}$$

[16] 'Sufficiently small' means here that the lower boundary $\eta > 0$ for consumption levels does not interfere with the optimal consumption levels pinned down by first-order conditions. Because these optimal consumption levels are strictly greater than zero, we can always find such $\eta > 0$. The role of this lower boundary is to ensure that longer consumption streams are preferred to smaller consumption streams, i.e., life is better than death. This is crucial for the ranking of consumption streams for a CEU decision maker but would be irrelevant for a standard expected utility decision maker (cf. below).

Proposition 6. *Under Assumption 1, the CEU (37) of the h-old agent from consumption plan $c = (c_h, ..., c_T)$ is equivalently given as*

$$CEU_h(\mathbf{c}) = \sum_{t=h}^{T} \rho_{h,t} u(c_t)$$

such that the effective discount factors $\rho^h = \left(\rho_{h,t}, ..., \rho_{h,T}\right)$ of the h-old agent are defined as

$$\rho_{h,t} = \beta_{h,t} \nu_{h,t}.$$

Finally, we have to ensure that the iso-elastic power per period utility functions of our model are consistent with the ranking condition (38). To this purpose, we consider period-utility functions $u : [\eta, \infty) \to \mathbb{R}_{\geq 0}$ given as

$$u(c) = \chi + \begin{cases} \frac{c^{1-\theta}}{1-\theta} & \text{for } \theta \neq 1 \\ \ln(c) & \text{for } \theta = 1 \end{cases} \tag{39}$$

such that the normalizing constant $\chi \geq 0$ has to ensure that $u(\eta) \geq 0$. If $u(c) < 0$, the ranking condition (38), which is crucial to the definition of CEU, could be violated. For $\theta < 1$ the period utility function is positive so that χ can be set to zero. For $\theta > 1$ we can set $\chi = -\frac{\eta^{1-\theta}}{1-\theta}$ and for $\theta = 1$ we can set $\chi = -\ln(\eta)$ to obtain, respectively, $u(\eta) = 0$. This explains the role of the lower boundary $\eta > 0$.

B Mathematical Proofs

B.1 Proof One of Theorem 1: The Linearity of Consumption Rule Argument

Our first proof of Theorem 1 establishes for a concavity parameter θ less than one that the 0-old sophisticated agent receives a greater marginal utility from instantly consuming Δw above the naive's consumption level than from handing down Δw for future consumption. The sophisticated agent's situation is reversed if the concavity parameter θ is greater than one. Because the comparison of absolute consumption levels at age 0 is, for a linear consumption rule, formally equivalent to the comparison of MPCs, this proof for the 0-old agent is sufficient to prove Theorem 1 for all h. Key to the proof is a forward induction argument that establishes the impact of an instantaneous consumption change of Δw on all future periods for arbitrary T.

Proof One of Theorem 1. Part (i). We show that $\theta < 1$ implies $m_h^n \leq m_h^s$.

Step 0. By linearity of the consumption rule—according to which both MPCs m_h^n and m_h^s are independent of the respective wealth levels w_h^n and w_h^s—Part (i) of Theorem 1 is proved if we can show that

$$m_h^n \leq m_h^s \qquad \Leftrightarrow \qquad m_h^n x \leq m_h^s x$$

for arbitrary $x > 0$. Without any loss of generality, set $h = 0$. Next, by setting $x = w_0$, we have $m_0^n \leq m_0^s$ if and only if

$$m_0^n w_0 \leq m_0^s w_0 \qquad \Leftrightarrow \qquad c_0^n \leq c_0^s.$$

We prove part (i) of Theorem 1 by showing, in the remainder of the proof, that $\theta < 1$ implies $c_0^n \leq c_0^s$.

Step 1. Consider, at first, the consumption profile $(c_0^n, \hat{c}_1^s, ..., \hat{c}_T^s)$—with corresponding wealth profile $(w_1^n, \hat{w}_2^s, ..., \hat{w}_T^s)$—according to which the 0-old sophisticated agent chooses (possibly suboptimally) the same consumption as the 0-old naive agent whereas all subsequent agents $t = 1, .., T$ choose the solution to the sophisticated problem starting at $t = 1$ with initial wealth level $w_1^n = w_0 - c_0^n$. Note that $\sum_{t=1}^{T} \hat{c}_t^s = w_1^n$. Next, let $\triangle w \in [0, c_0^n)$

and fix $c_0^n - \triangle w$ for some $\triangle w > 0$ as the 0-old agent's modified choice. Denote by $\left(\hat{c}_1^s\left[\triangle w\right], ..., \hat{c}_T^s\left[\triangle w\right]\right)$ the solution of this modified sophisticated problem for the periods $t = 1, ..., T$ with corresponding modified wealth profile $\left(w_1^n\left[\triangle w\right], \hat{w}_2^s\left[\triangle w\right], ..., \right)$. Note that $\sum_{t=1}^{T} \hat{c}_t^s\left[\triangle w\right] = w_1^n\left[\triangle w\right] = w_1^n + \triangle w$.

Our claim is proved if we can establish that $\theta < 1$ implies, for all $\triangle w \in (0, c_0^n)$,

$$U_0\left(c_0^n - \triangle w, \hat{c}_1^s\left[\triangle w\right], ..., \hat{c}_T^s\left[\triangle w\right]\right) < U_0\left(c_0^n, \hat{c}_1^s, ..., \hat{c}_T^s\right). \tag{40}$$

In words: If (40) holds, the sophisticated 0-old agent would never consume strictly less than her naive 0-old counterpart. Or put differently: Even if c_0^n is a suboptimal choice, the sophisticated agent would do strictly worse if she chooses instead $c_0^n - \triangle w$ with $\triangle w > 0$.

Step 2. Given linearity of the consumption rule in wealth levels, we know that consumption in future periods will be increased in proportion to current consumption. That is, consumption in period $t = 1, ..., T$ increases by $\frac{\hat{c}_t^s}{w_1^n} \triangle w$. To make this argument precise, observe that we obtain for the 1-old agent, by linearity of the consumption rule,

$$\hat{c}_1^s\left[\triangle w\right] = m_1^s w_1^n\left[\triangle w\right] = m_1^s\left(w_1^n + \triangle w\right) = \hat{c}_1^s + \frac{\hat{c}_1^s}{w_1^n} \triangle w.$$

Next turn to the 2-old agent and observe that

$$\begin{aligned}
\hat{c}_2^s\left[\triangle w\right] &= m_2^s \hat{w}_2^s\left[\triangle w\right] \\
&= m_2^s\left(w_1^n + \triangle w - \hat{c}_1^s\left[\triangle w\right]\right) \\
&= m_2^s\left(w_1^n - \hat{c}_1^s\right) + \frac{\hat{c}_2^s}{\hat{w}_2^s}\left(\triangle w - \frac{\hat{c}_1^s}{w_1^n} \triangle w\right) \\
&= m_2^s \hat{w}_2^s + \frac{\hat{c}_2^s}{\hat{w}_2^s}\left(\frac{w_1^n}{w_1^n} - \frac{\hat{c}_1^s}{w_1^n}\right) \triangle w \\
&= \hat{c}_2^s + \frac{\hat{c}_2^s}{w_1^n} \triangle w,
\end{aligned}$$

where the last step follows from $\hat{w}_2^s = w_1^n - \hat{c}_1^s$. More generally, given $\hat{c}_t^s\left[\triangle w\right] = \hat{c}_t^s + \frac{\hat{c}_t^s}{w_1^n} \triangle w$

for all $t < h$ we have for h

$$
\begin{aligned}
\hat{c}_h^s [\triangle w] &= m_h^s \hat{w}_h^s [\triangle w] \\
&= m_h^s \left(w_1^n + \triangle w - \sum_{t=1}^{h-1} \hat{c}_t^s [\triangle w] \right) \\
&= m_h^s \left(w_1^n - \sum_{t=1}^{h-1} \hat{c}_t^s \right) + \frac{\hat{c}_h^s}{\hat{w}_h^s} \left(1 - \sum_{t=1}^{h-1} \frac{\hat{c}_t^s}{w_1^n} \right) \triangle w \\
&= m_h^s \hat{w}_h^s + \frac{\hat{c}_h^s}{\hat{w}_h^s} \left(\frac{w_1^n}{w_1^n} - \sum_{t=1}^{h-1} \frac{\hat{c}_t^s}{w_1^n} \right) \triangle w \\
&= \hat{c}_h^s + \frac{\hat{c}_h^s}{w_1^n} \triangle w,
\end{aligned}
$$

where the last step follows from $\hat{w}_h^s = w_1^n - \sum_{t=1}^{h-1} \hat{c}_t^s$. By the above induction argument, inequality (40) is therefore equivalently given as

$$
U_0 \left(c_0^n - \triangle w, \hat{c}_1^s \left(1 + \frac{\triangle w}{w_1^n} \right), ..., \hat{c}_T^s \left(1 + \frac{\triangle w}{w_1^n} \right) \right) < U_0 \left(c_0^n, \hat{c}_1^s, ..., \hat{c}_T^s \right). \tag{41}
$$

Step 3. Rewrite inequality (41) as

$$
U_0 (x_0 + h) < U_0 (x_0) \tag{42}
$$

such that

$$
\begin{aligned}
x_0 &= \left(c_0^n, \hat{c}_1^s, ..., \hat{c}_T^s \right), \\
h &= \left(- \triangle w, \hat{c}_1^s \frac{\triangle w}{w_1^n}, ..., \hat{c}_T^s \frac{\triangle w}{w_1^n} \right).
\end{aligned}
$$

Recall from Taylor-series approximation theory that

$$
U_0 (x_0 + h) = U_0 (x_0) + \frac{dU_0}{dx} (x_0) \, h + R (\triangle w)
$$

whereby the residual term vanishes fast:

$$
\lim_{\triangle w \to 0} \frac{R (\triangle w)}{\triangle w} = 0. \tag{43}
$$

For $\triangle w > 0$ we can thus equivalently transform (42) to

$$\begin{aligned}
\frac{U_0(x_0+h)}{\triangle w} &< \frac{U_0(x_0)}{\triangle w} \\
&\Leftrightarrow \\
\frac{U_0(x_0)}{\triangle w} + \frac{1}{\triangle w}\frac{dU_0}{dx}(x_0)\,h + \frac{R(\triangle w)}{\triangle w} &< \frac{U_0(x_0)}{\triangle w} \\
&\Leftrightarrow \\
\frac{1}{\triangle w}\frac{dU_0}{dx}(x_0)\,h + \frac{R(\triangle w)}{\triangle w} &< 0.
\end{aligned}$$

Taking the limit of the l.h.s. gives us, by (43),

$$\lim_{\triangle w \to 0} \frac{1}{\triangle w}\frac{dU_0}{dx}(x_0)\,h + \frac{R(\triangle w)}{\triangle w} = \lim_{\triangle w \to 0} \frac{1}{\triangle w}\frac{dU_0}{dx}(x_0)\,h.$$

By continuity of the utility function, we obtain that

$$\begin{aligned}
U_0(x_0+h) &< U_0(x_0) \\
&\Leftrightarrow \\
U_0\left(c_0^n - \triangle w, \hat{c}_1^s\left(1+\frac{\triangle w}{w_1^n}\right), ..., \hat{c}_T^s\left(1+\frac{\triangle w}{w_1^n}\right)\right) &< U_0\left(c_0^n, \hat{c}_1^s, ..., \hat{c}_T^s\right)
\end{aligned}$$

for sufficiently small $\triangle w > 0$ if and only if

$$\begin{aligned}
\lim_{\triangle w \to 0} \frac{1}{\triangle w}\frac{dU_0}{dx}(x_0)\,h &< 0 \\
&\Leftrightarrow \\
\lim_{\triangle w \to 0} \frac{1}{\triangle w}\left(\left(c_0^n\right)^{-\theta}(-)\triangle w + \sum_{t=1}^{T}\rho_{0,t}\left(\hat{c}_t^s\right)^{-\theta}\hat{c}_t^s\frac{\triangle w}{w_1^n}\right) &< 0 \\
&\Leftrightarrow \\
\frac{1}{w_1^n}\sum_{t=1}^{T}\rho_{0,t}\left(\hat{c}_t^s\right)^{-\theta}\hat{c}_t^s &< \left(c_0^n\right)^{-\theta}. \qquad (44)
\end{aligned}$$

Step 4. We are going to show that inequality (44) holds for $\theta < 1$. At first, transform the l.h.s. of (44) as follows

$$\begin{aligned}\frac{1}{w_1^n}\sum_{t=1}^{T}\rho_{0,t}\left(\hat{c}_t^s\right)^{-\theta}\hat{c}_t^s &= \frac{1}{w_1^n}\sum_{t=1}^{T}\rho_{0,t}\left(\hat{c}_t^s\right)^{1-\theta}\\ &= \frac{1}{w_1^n}\sum_{t=1}^{T}\rho_{0,t}\left(c_t^n\right)^{1-\theta}\left(\frac{\hat{c}_t^s}{c_t^n}\right)^{1-\theta}\\ &= \frac{1}{w_1^n}\sum_{t=1}^{T}\left(c_0^n\right)^{-\theta}c_t^n\left(\frac{\hat{c}_t^s}{c_t^n}\right)^{1-\theta}\\ &= \frac{1}{w_1^n}\left(c_0^n\right)^{-\theta}\sum_{t=1}^{T}c_t^n\left(\frac{\hat{c}_t^s}{c_t^n}\right)^{1-\theta},\end{aligned}$$

where the third line follows from the first-order condition of the naive agent implying $c_0^{n^{-\theta}}=\rho_{0,t}c_t^{n^{-\theta}}$. Next, consider the function

$$f\left(c_1,...,c_T\right)=\sum_{t=1}^{T}c_t^n\left(\frac{c_t}{c_t^n}\right)^{1-\theta},$$

where $\{c_h^n\}_{h=0}^T$ are parameters, and recall that $\sum_{t=1}^T c_t^n=w_1^n$. The shape of this function depends on the parameter θ. Taking the derivatives w.r.t. c_t we get

$$f'=(1-\theta)\left(\frac{c_t}{c_t^n}\right)^{-\theta}\begin{cases}>0 & \text{for } \theta<1\\ <0 & \text{for } \theta>1\end{cases}$$

and

$$f''=-\theta\left(1-\theta\right)\left(\frac{c_t}{c_t^n}\right)^{-(1+\theta)}\frac{1}{c_t^n}\begin{cases}<0 & \text{for } \theta<1\\ >0 & \text{for } \theta>1.\end{cases}$$

These derivatives show that the function $f(\cdot)$ is strictly increasing and strictly concave for $\theta<1$ whereas it is strictly decreasing and strictly convex for $\theta>1$. For $\theta<1$ the constrained maximization problem—resulting in a unique maximizer—is

$$\max f(c_1,...,c_T)\qquad \text{s.t.}\sum_{t=1}^{T}c_t=w_1^n.$$

The Lagrangian is

$$L=f(c_1,...,c_T)-\lambda\left(\sum_{t=1}^{T}c_t-w_1^n\right)$$

with the first order condition for each $t = 1, ..., T$

$$\frac{\partial L}{\partial c_t} = (1-\theta)\left(\frac{c_t}{c_t^n}\right)^{-\theta} - \lambda = 0.$$

Combining the first-order conditions yields $\frac{c_{t+1}}{c_t} = \frac{c_{t+1}^n}{c_t^n}$ so that f achieves its unique maximum at $(c_1, ..., c_T) = (c_1^n, ..., c_T^n)$. Consequently, we have

$$\frac{1}{w_1^n}\sum_{t=1}^{T}\rho_{0,t}\left(\hat{c}_t^s\right)^{-\theta}\hat{c}_t^s < \frac{1}{w_1^n}\left(c_0^n\right)^{-\theta}\sum_{t=1}^{T}c_t^n\left(\frac{c_t^n}{c_t^n}\right)^{1-\theta} = \left(c_0^n\right)^{-\theta}$$

whenever $(\hat{c}_1^s, ..., \hat{c}_T^s) \neq (c_1^n, ..., c_T^n)$. This shows that $\theta < 1$ implies inequality (44).

Step 5. Through Steps 3 and 4 we have established that $\theta < 1$ implies the existence of some sufficiently small $\varepsilon > 0$ such that

$$\begin{array}{rcl} U_0\left(x_0 + h\right) & < & U_0\left(x_0\right) \\ & \Leftrightarrow & \\ U_0\left(c_0^n - \triangle w, \hat{c}_1^s\left(1 + \frac{\triangle w}{w_1^n}\right), ..., \hat{c}_T^s\left(1 + \frac{\triangle w}{w_1^n}\right)\right) & < & U_0\left(c_0^n, \hat{c}_1^s, ..., \hat{c}_T^s\right) \end{array}$$

for all $\triangle w \in (0, \varepsilon)$. That is, $U_0(\cdot)$ takes on a unique local maximum—i.e., over all $\triangle w \in [0, \varepsilon)$—at $\triangle w = 0$ whenever $\theta < 1$. To prove our claim, it remains to be shown that $U_0(\cdot)$ also takes on a global maximum—i.e., over all $\triangle w \in [0, c_0^n)$—at $\triangle w = 0$ whenever $\theta < 1$. To see this, note that any critical point $\triangle w^* \in [0, c_0^n)$ must satisfy

$$\begin{array}{rcl} \left.\frac{dU_0(\cdot)}{d \triangle w}\right|_{\triangle w^*} & = & 0 \\ & \Leftrightarrow & \\ \sum_{t=1}^{T}\rho_{0,t}\left(\hat{c}_t^s\left(1 + \frac{\triangle w^*}{w_1^n}\right)\right)^{-\theta}\frac{\hat{c}_t^s}{w_1^n} & = & \left(c_0^n - \triangle w^*\right)^{-\theta}. \end{array}$$

But because the l.h.s. of this equation is strictly decreasing in $\triangle w$ whereas the r.h.s. is strictly increasing in $\triangle w$, there can exist at most one critical point $\triangle w^*$ on $[0, c_0^n)$. By this single-crossing argument, $\triangle w = 0$ must also be the global maximum if it is a local maximum.

This concludes the proof of Part (i).□

Proof of Theorem 1. Part (ii). We show that $\theta > 1$ implies $c_h^n \geq c_h^s$ for $h = 0$. Our proof is a mirrored version of the proof of Part (i), where we combine and shorten a few steps.

Steps 1-2. Let $\triangle w \in [0, w_1^n)$ and fix $c_0^n + \triangle w$ for some $\triangle w > 0$ as the 0-old agent's choice. Our claim is proved if we can establish that $\theta > 1$ implies, for all $\triangle w \in (0, w_1^n)$,

$$U_0\left(c_0^n + \triangle w, \hat{c}_1^s\left(1 - \frac{\triangle w}{w_1^n}\right), ..., \hat{c}_T^s\left(1 - \frac{\triangle w}{w_1^n}\right)\right) < U_0\left(c_0^n, \hat{c}_1^s, ..., \hat{c}_T^s\right). \tag{45}$$

In words: If (45) holds, the sophisticated 0-old agent would never consume strictly more than her naive 0-old counterpart.

Step 3. Let

$$\hat{h} = \left(\triangle w, -\hat{c}_1^s \frac{\triangle w}{w_1^n}, ..., -\hat{c}_T^s \frac{\triangle w}{w_1^n}\right).$$

In analogy to the proof of Part (i), we have that

$$\begin{aligned} U_0\left(x_0 + \hat{h}\right) &< U_0\left(x_0\right) \\ &\Leftrightarrow \\ U_0\left(c_0^n + \triangle w, \hat{c}_1^s\left(1 - \frac{\triangle w}{w_1^n}\right), ..., \hat{c}_T^s\left(1 - \frac{\triangle w}{w_1^n}\right)\right) &< U_0\left(c_0^n, \hat{c}_1^s, ..., \hat{c}_T^s\right) \end{aligned}$$

for sufficiently small $\triangle w > 0$ if and only if

$$\begin{aligned} \lim_{\triangle w \to 0} \frac{1}{\triangle w} \frac{dU_0}{dx}\left(x_0\right)\hat{h} &< 0 \\ &\Leftrightarrow \\ \lim_{\triangle w \to 0} \frac{1}{\triangle w}\left(\left(c_0^n\right)^{-\theta} \triangle w - \sum_{t=1}^{T} \rho_{0,t}\left(\hat{c}_t^s\right)^{-\theta} \hat{c}_t^s \frac{\triangle w}{w_1^n}\right) &< 0 \\ &\Leftrightarrow \\ \left(c_0^n\right)^{-\theta} &< \frac{1}{w_1^n} \sum_{t=1}^{T} \rho_{0,t}\left(\hat{c}_t^s\right)^{-\theta} \hat{c}_t^s \\ &\Leftrightarrow \\ \left(c_0^n\right)^{-\theta} &< \frac{1}{w_1^n} \sum_{t=1}^{T} \rho_{0,t}\left(\hat{c}_t^s\right)^{-\theta} \hat{c}_t^s. \end{aligned}$$

Steps 4-5. Recall from Step 4 of Part (i) that the function is for $\theta > 1$ strictly decreasing and strictly convex. Consequently, there exists a unique minimum at $(c_1^n, ..., c_T^n)$ implying

$$\frac{1}{w_1^n}\left(c_0^n\right)^{-\theta} \sum_{t=1}^{T} c_t^n \left(\frac{c_t^n}{c_t^n}\right)^{1-\theta} = \left(c_0^n\right)^{-\theta} < \frac{1}{w_1^n} \sum_{t=1}^{T} \rho_{0,t}\left(\hat{c}_t^s\right)^{-\theta} \hat{c}_t^s$$

whenever $(\hat{c}_1^s, ..., \hat{c}_T^s) \neq (c_1^n, ..., c_T^n)$. This establishes that $\theta > 1$ implies inequality (45) for sufficiently small $\triangle w > 0$. By the same argument as in Step 5 of Part (i), this local minimizer is also the global minimizer so that (45) holds for all $\triangle w \in (0, w_1^n)$.□□

B.2 Proof Two of Theorem 1: The Backward Induction Argument

Our second proof of Theorem 1 is based on the recursive presentations of the marginal propensities to consume of the sophisticated and the naive agent. The different implications for the cases $\theta < 1$ versus $\theta > 1$ result from a simple application of Jensen's inequality to strictly concave and strictly convex functions, respectively. Because the proof of Theorem 1 will be implied by the proof of Lemma 1, we prove, at first, Lemma 1.

Proof of Lemma 1. Part (i): We show for $h \in \{0, \ldots, T-2\}$:

(i) $\theta < 1$ *implies* $m_h^n = m_h^s$ *if* $m_t^{n,h} = m_t^s$ *for all* $t \geq h+1$.

(ii) $\theta < 1$ *implies* $m_h^n < m_h^s$ *if* $m_t^{n,h} \neq m_t^s$ *for some* $t \geq h+1$.

Recall from (5) and (7) the following expressions for MPCs

$$m_h^s = \frac{1}{1 + \left(\rho_{h,h+1}\zeta_{h+1}^h\right)^{\frac{1}{\theta}}}$$

where

$$\zeta_t^h = m_t^{s^{1-\theta}} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \left(1 - m_t^s\right)^{1-\theta} \zeta_{t+1}^h \tag{46}$$

as well as

$$m_t^{n,h} = \frac{1}{1 + \left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\right)^{\frac{1}{\theta}} m_{t+1}^{n,h^{-1}}}. \tag{47}$$

Using these expressions gives us at age $t = h$

$$\begin{aligned} m_h^n &\leq m_h^s \\ &\Leftrightarrow \\ \left(\rho_{h,h+1}\zeta_{h+1}^h\right)^{\frac{1}{\theta}} &\leq \left(\frac{\rho_{h,h+1}}{\rho_{h,h}}\right)^{\frac{1}{\theta}} m_{h+1}^{n,h^{-1}} \\ &\Leftrightarrow \\ m_{h+1}^{n,h^{\theta}}\zeta_{h+1}^h &\leq 1. \end{aligned}$$

Next, we appropriately transform ζ_t^h. To this purpose, notice from (47) that

$$\frac{\rho_{h,t+1}}{\rho_{h,t}} = \left(\frac{1-m_t^{n,h}}{m_t^{n,h}}\right)^{\theta} m_{t+1}^{n,h^\theta}.$$

Using this in (46) we get recursively for $t = T-2, \ldots, h+1$

$$\zeta_t^h = m_t^{s^{1-\theta}} + \left(\frac{1-m_t^{n,h}}{m_t^{n,h}}\right)^{\theta} (1-m_t^s)^{1-\theta}\, m_{t+1}^{n,h^\theta}\, \zeta_{t+1}^h$$

$$\Leftrightarrow \qquad m_t^{n,h^\theta} \zeta_t^h = \left(\frac{m_t^{n,h}}{m_t^s}\right)^{\theta} m_t^s + \left(\frac{1-m_t^{n,h}}{1-m_t^s}\right)^{\theta} (1-m_t^s)\, m_{t+1}^{n,h^\theta}\, \zeta_{t+1}^h. \tag{48}$$

The remainder of the proof proceeds by backward induction on (48) over $t = T-1, \ldots, h+1$.

Claims: Firstly, we claim that, for all $t \in \{h+1, \ldots, T-1\}$, $\theta < 1$ implies

$$m_t^{n,h^\theta} \zeta_t^h = 1 \tag{49}$$

if $m_t^{n,h} = m_t^s$ for all $t \geq h+1$.

Secondly, we claim that, for all $t \in \{h+1, \ldots, T-1\}$, $\theta < 1$ implies

$$m_t^{n,h^\theta} \zeta_t^h < 1 \tag{50}$$

if $m_t^{n,h} \neq m_t^s$ for some $t \geq h+1$.

Base Case: Recall that $m_T^n = m_T^{n,h} = m_T^s = 1$. In period $t = T-1$ we have

$$m_{T-1}^{n,h^\theta} \zeta_{T-1}^h = \left(\frac{m_{T-1}^{n,h}}{m_{T-1}^s}\right)^{\theta} m_{T-1}^s + \left(\frac{1-m_{T-1}^{n,h}}{1-m_{T-1}^s}\right)^{\theta} \left(1-m_{T-1}^s\right).$$

Suppose, at first, that $m_{T-1}^{n,h} = m_{T-1}^s$. Then our first claim (49) is trivially satisfied for $t = T-1$ because of

$$m_t^{n,h^\theta} \zeta_t^h = 1$$

irrespective of the value of θ.

Suppose now that $m^{n,h}_{T-1} \neq m^s_{T-1}$, implying

$$\frac{m^{n,h}_{T-1}}{m^s_{T-1}} \neq \frac{1 - m^{n,h}_{T-1}}{1 - m^s_{T-1}}.$$

By the strict version of Jensen's inequality, we obtain for $\theta < 1$

$$\begin{aligned} m^{n,h^\theta}_{T-1} \zeta^h_{T-1} &= \left(\frac{m^{n,h}_{T-1}}{m^s_{T-1}} \right)^\theta m^s_{T-1} + \left(\frac{1 - m^{n,h}_{T-1}}{1 - m^s_{T-1}} \right)^\theta \left(1 - m^s_{T-1}\right) \\ &< \left(\left(\frac{m^{n,h}_{T-1}}{m^s_{T-1}} \right) m^s_{T-1} + \left(\frac{1 - m^{n,h}_{T-1}}{1 - m^s_{T-1}} \right) \left(1 - m^s_{T-1}\right) \right)^\theta \\ &= 1 \end{aligned}$$

because x^θ is strictly concave for $\theta < 1$. Consequently, our second claim (50) is satisfied for $t = T - 1$.

Backward Induction Step: Suppose that the first claim (49) has been proved for period $i + 1$. That is, we have shown that $\theta < 1$ implies

$$m^{n,h^\theta}_{i+1} \zeta^h_{i+1} = 1 \tag{51}$$

if $m^{n,h}_t = m^s_t$ for all $t \geq i + 1$. Rewrite (48) as

$$m^{n,h^\theta}_i \zeta^h_i = \underbrace{\left(\frac{m^{n,h}_i}{m^s_i} \right)^\theta m^s_i + \left(\frac{1 - m^{n,h}_i}{1 - m^s_i} \right)^\theta \left(1 - m^s_i\right)}_{=\Lambda(m^{n,h}_i, m^s_i)} m^{n,h^\theta}_{i+1} \zeta^h_{i+1}.$$

By the same reasoning as in the base case, we have that $\theta < 1$ implies

$$\Lambda(m^{n,h}_i, m^s_i) \leq 1 \tag{52}$$

whereby this inequality is strict if and only if $m^{h,n}_i \neq m^s_i$. Since

$$x + y \leq 1 \text{ and } b \leq 1 \text{ implies } x + by \leq 1,$$

(51) together with (52) gives us the desired result that $\theta < 1$ implies

$$m^{n,h^\theta}_i \zeta^h_i = 1 \tag{53}$$

if $m_i^{h,n} = m_i^s$ whereas we have

$$m_i^{n,h^\theta} \zeta_i^h < 1$$

if $m_i^{h,n} \neq m_i^s$.

Next suppose that we have proved the second claim (50) for period $i+1$. That is, we have shown that $\theta < 1$ implies

$$m_{i+1}^{n,h^\theta} \zeta_{i+1}^h < 1$$

if $m_t^{n,h} \neq m_t^s$ for some $t \geq i+1$. Because of (52), we must have that

$$m_i^{n,h^\theta} \zeta_i^h < 1.$$

Combining both cases proves Part (i) of Lemma 1.□

Proof of Lemma 1. Part (ii): We show for $h \in \{0, \ldots, T-2\}$:

(i) $\theta > 1$ *implies* $m_h^n = m_h^s$ *if* $m_t^{n,h} = m_t^s$ *for all* $t \geq h+1$.

(ii) $\theta > 1$ *implies* $m_h^n < m_h^s$ *if* $m_t^{n,h} \neq m_t^s$ *for some* $t \geq h+1$.

The proof proceeds exactly as the proof of Part (i) of Lemma 1 whereby we prove the following two claims:

Firstly, for all $t \in \{h+1, \ldots, T-1\}$, $\theta > 1$ implies

$$m_t^{n,h^\theta} \zeta_t^h = 1$$

if $m_t^{n,h} = m_t^s$ for all $t \geq h+1$.

Secondly, for all $t \in \{h+1, \ldots, T-1\}$, $\theta > 1$ implies

$$m_t^{n,h^\theta} \zeta_t^h > 1 \tag{54}$$

if $m_t^{n,h} \neq m_t^s$ for some $t \geq h+1$.

The only difference to the proof of Part (i) is the reversed strict inequality in claim (54) which follows, by the strict version of Jensen's inequality, by strict convexity of x^θ for $\theta > 1$.□□

Proof Two of Theorem 1. To prove Part (i), we have to show that $\theta < 1$ implies $m_h^n \leq m_h^s$. Recall from the proof of Lemma 1(i) that

$$m_t^{n,h^\theta} \zeta_t^h \leq 1 \text{ for all } t \in \{T-2, \ldots, h+1\} \text{ implies } m_h^n \leq m_h^s.$$

Moreover, the proof of Lemma 1(i) had established that $\theta < 1$ implies either $m_t^{n,h^\theta} \zeta_t^h = 1$ or $m_t^{n,h^\theta} \zeta_t^h < 1$ for all $t \in \{T-2, \ldots, h+1\}$. An analogous argument applies to Part (ii) of Theorem 1.□□

B.3 Proof of Proposition 5

Our proof of Proposition 5 is based on recursive methods.

Sophisticated Agent. Our proof is by backward induction.

Claims: The value function of the sophisticated agent in any period $t \geq h$ is given by

$$U_t^h(w_t) = \frac{1}{1-\theta} \zeta_t^h w_t^{1-\theta} \tag{55}$$

with associated policy function

$$c_h^s = m_h^s w_h. \tag{56}$$

Base case: In period T we have $c_T^s = w_T$ and thus $U_T^h = \frac{1}{1-\theta} w_T^{1-\theta}$ and $m_T^s = 1$.

Backward Induction Steps: Suppose the claims (55) and (56) have been shown for all periods $h+1, \ldots, T$. Then iterate backward for all $t = T-1, \ldots, h+1$ using (55) in (20) to get, also using resource constraint (25),

$$\begin{aligned} U_t^h &= u(c_t) + \frac{\rho_{h,t+1}}{\rho_{h,t}} \frac{1}{1-\theta} \left(\mathbb{E} \left[\left((1-\theta) U_{t+1}^h \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \\ &= \frac{1}{1-\theta} \left(u(c_t^s) + \frac{\rho_{h,t+1}}{\rho_{h,t}} \zeta_{t+1}^h \left(\mathbb{E} \left[\left(w_{t+1}^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \right) \\ &= \frac{1}{1-\theta} \left((m_t^s)^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}} (1-m_t^s)^{1-\theta} \zeta_{t+1}^h \Theta \left(\hat{\alpha}_t^*, R^f, R_{t+1}, \pi \right) \right) w_t^{1-\theta} \\ &= \frac{1}{1-\theta} \zeta_t^h w_t^{1-\theta}, \end{aligned} \tag{57}$$

which defines (29)—for an age varying optimal portfolio share $\hat{\alpha}_t^\star$, constancy of which is proven below—and establishes the backward recursion of ζ_t^h in (28).

Next, in period h use (55) in (20) to get

$$U_h^h = \frac{1}{1-\theta} \max_{c_h^s, w_{h+1}, \hat{\alpha}_h^s} \left\{ {c_h^s}^{1-\theta} + \rho_{h,h+1}\zeta_{h+1}^h \left(\mathbb{E}\left[\left(w_{h+1}^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \right\}. \tag{58}$$

Use the resource constraint (25) in the above to obtain, by the separation between the optimal consumption and the optimal portfolio choice,

$$U_h^h = \frac{1}{1-\theta} \max_{c_h^s} \left\{ {c_h^s}^{1-\theta} + \rho_{h,h+1} \left(w_h - c_h^s \right)^{1-\theta} \right\} \zeta_{h+1}^h \underbrace{\max_{\hat{\alpha}_t} \left\{ \left(\mathbb{E}\left[\left(R_{h+1}^p (\hat{\alpha}_h)^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \right\}}_{=\Theta\left(\hat{\alpha}_h^*, \zeta_{h+1}^h, R^f, R_{h+1}, \pi\right)}$$

with first-order condition for c_h^s

$${c_h^s}^{-\theta} - \rho_{h,h+1} \left(w_h - c_h \right)^{-\theta} \zeta_{h+1}^h \Theta\left(\hat{\alpha}_h^*, R^f, R_{h+1}, \pi \right) = 0,$$

where $\hat{\alpha}_h^*$ is the optimal portfolio share further characterized below. We thus get

$${c_h^s}^* = m_h^s w_t$$

where

$$m_h^s = \frac{1}{1 + \left[\rho_{h,h+1}\zeta_{h+1}^h \Theta\left(\hat{\alpha}_h^*, R^f, R_{h+1}, \pi \right) \right]^{\frac{1}{\theta}}}.$$

which is (58) and proves the claims.

Naive Agent. For the naive agent, we essentially follow the same steps with the following modifications:

- The maximization problem in (58) is solved for all $t = h, \ldots, T-1$, thus

$$U_t^{n,h} = \frac{1}{1-\theta} \max_{c_t^{n,h}, w_{t+1}, \hat{\alpha}_t^{n,h}} \left\{ {c_t^{n,h}}^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \zeta_{t+1}^h \left(\mathbb{E}\left[\left(w_{t+1}^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \right\},$$

  which, using the resource constraint and the separation between the optimal consumption and the portfolio choice, gives

$$m_t^{n,h} = \frac{1}{1 + \left[\frac{\rho_{h,t+1}}{\rho_{h,t}} \zeta_{t+1}^h \Theta\left(\hat{\alpha}_t^*, R^f, R_{t+1}, \pi \right) \right]^{\frac{1}{\theta}}}. \tag{59}$$

- Using the solution back in the value function as in (57) gives

$$
\begin{aligned}
U_t^{n,h} &= \frac{1}{1-\theta}\left(\left(m_t^{n,h}\right)^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}}\zeta_{t+1}^{n,h}\left(1-m_t^{n,h}\right)^{1-\theta}\Theta\left(\hat{\alpha}_t^{n,h^*}, R^f, R_{t+1}, \pi\right)\right) w_t^{1-\theta} \\
&= \frac{1}{1-\theta}\left(\left(m_t^{n,h}\right)^{1-\theta} + \left(1-m_t^{n,h}\right)^{1-\theta}\left(\frac{1-m_t^{n,h}}{m_t^{n,h}}\right)^{\theta}\right) w_t^{1-\theta} \\
&= \frac{1}{1-\theta} m_t^{n,h^{-\theta}} w_t^{1-\theta}.
\end{aligned}
$$

- We thus find $\zeta_t^h = m_t^{h^{-\theta}}$. Using this in (59) we finally obtain

$$
m_t^h = \frac{1}{1 + \left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\Theta\left(\hat{\alpha}_t^{n,h^*}, R^f, R_{t+1}, \pi\right)\right)^{\frac{1}{\theta}} m_{t+1}^{n,h^{-1}}}.
$$

Optimal Portfolio Choice. Since $\Theta\left(\hat{\alpha}_t, R^f, R_{t+1}, \pi\right)$ is the same for both agents and by the i.i.d. assumption on R_{t+1} we obtain $\hat{\alpha}_t^{s^*} = \hat{\alpha}_t^{n,h^*} = \hat{\alpha}^*$, where from the first-order condition of the optimal portfolio allocation problem $\hat{\alpha}^*$ is the solution to

$$
\mathbb{E}\left[R_{t+1}^p(\hat{\alpha}_t)^{-\sigma}\right] = \int R_{t+1}^p(\hat{\alpha}_t)^{-\sigma} d\pi = 0
$$

and thus the optimal portfolio allocation problem is a static decision problem, which is parameterized by risk aversion σ. □□

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