Supplementary Appendix

(Not for Publication)

"Higher-Order Income Risk Over the Business Cycle"

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S.A Fitting Moments of the FGLD

This supplementary appendix describes how we fit the Flexible Generalized Lambda Distribution (FGLD). The quantile function is

$$Q(p;\lambda) = F^{-1}(p;\lambda) = x = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{p^{\lambda_3}-1}{\lambda_3} - \frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right) \tag{S.A.1}$$

where λ_1 is a location and λ_2 is a scale parameter, λ_3, λ_4 in turn are tail index parameters.[1]

We will need to use the relationship between the quantile function and the probability density function (PDF). Noticing that $x = F^{-1}(p) = Q(p)$ and $F(x) = p$ we can derive the PDF $f(x)$ from the quantile function $Q(p)$ by

$$f(x) = f(Q(p)) = \frac{\partial F(x)}{\partial x} = \frac{\partial p}{\partial Q(p)} = \frac{1}{\frac{\partial Q(p)}{\partial p}}. \tag{S.A.2}$$

Differentiating (S.A.1) we therefore find the PDF to be

$$f(x) = f(Q(p)) = \frac{\lambda_2}{p^{\lambda_3-1} + (1-p)^{\lambda_4-1}}. \tag{S.A.3}$$

Lakhany and Mausser (2000) and Su (2007) describe how to estimate the parameters of (S.A.1) using moments of the distribution. The kth raw moment of a random variable X is given as

$$E\left[X^k\right] = \int_{-\infty}^{\infty} x^k f(x)dx, \quad k \geq 1$$

where $f(x)$ is the distribution function. Setting $k = 1$ gives the expected value $\mu_1 = E[X]$.

[1] The parametric constraints are $\lambda_2 > 0$, and $\min\{\lambda_3, \lambda_4\} > -\frac{1}{4}$.

The kth central moment is defined as

$$E\left[(X-\mu_1)^k\right] = \int_{-\infty}^{\infty} (x-\mu_1)^k f(x)dx, \quad k \geq 1.$$

We can use binomial expansion to write central moments in terms of raw moments as

$$E\left[(X-\mu_1)^k\right] = E\left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j (X)^{k-j} \mu_1^j\right] \tag{S.A.4}$$

where $\binom{k}{j}$ are binomial coefficients.

Now apply the same logic to evaluate the kth raw moment of a percentile function. Use variable substitution $p = Q^{-1}(p) = F(x)$, noticing that $Q^{-1}(-\infty) = 0$ and $Q^{-1}(\infty) = 1$ so that the integration bounds change. Furthermore, use (S.A.2) giving $f(x) = \frac{dp}{dQ(p)}$ to rewrite

$$\int_{-\infty}^{\infty} x^k f(x)dx = \int_0^1 Q(p)^k \frac{dp}{dQ(p)} dQ(p) = \int_0^1 Q(p)^k dp. \tag{S.A.5}$$

Hence the kth raw moment using quantile functions is given by

$$E\left[X^k\right] = \int_0^1 Q(p)^k dp.$$

Next, observe that (S.A.1) can be rewritten as

$$\begin{aligned} Q(p) = F^{-1}(p) = x &= \lambda_1 - \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} + \frac{1}{\lambda_2}\left(\frac{p^{\lambda_3}}{\lambda_3} - \frac{(1-p)^{\lambda_4}}{\lambda_4}\right) \\ &= a + b\tilde{Q}(p). \end{aligned}$$

Let X be the random variable with quantile function $Q(p)$ and let Y be the random variable with quantile function $\tilde{Q}(p)$. We then have

$$\begin{aligned} E[X] &= a + bE[Y], \quad k = 1 \\ E\left[(X - E[X])^k\right] &= b^k E\left[(Y - E[Y])^k\right], \quad k > 1 \end{aligned}$$

for the kth central moments. In what follows, we denote the raw moments of Y by ν, hence $\nu_k = EY^k$. Using (S.A.4) we thus get for the first four central moments (recalling that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, with $\binom{n}{n} = \binom{n}{0} = 1$):

$$\begin{aligned}\mu_1 &= E[X] = a + bE[Y] = a + b\nu_1 \\ &= \lambda_1 - \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} + \frac{1}{\lambda_2}\nu_1.\end{aligned}$$

For the remaining moments, we rewrite (S.A.4) to get

$$\begin{aligned}E\left[(Y - E[Y])^k\right] &= E\left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j \, (Y)^{k-j} \, \nu(1)^j\right] \\ &= \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j \, E\left[(Y)^{k-j}\right] \nu(1)^j\right]\end{aligned}$$

We can therefore write explicitly

$$\begin{aligned}\mu_2 &= b^2\left(E[Y^2] - (E[Y])^2\right) = \frac{1}{\lambda_2^2}(\nu_2 - \nu_1^2) \\ \mu_3 &= b^3 E\left[\sum_{j=0}^{3} \binom{3}{j} (-1)^j \, (Y)^{3-j} \, (\nu_1)^j\right] \\ &= b^3 E\left[Y^3 - 3Y^2\nu_1 + 3Y\nu_1^2 - \nu_1^3\right] \\ &= \frac{1}{\lambda_2^3}\left(\nu_3 - 3\nu_1\nu_2 + 2\nu_1^3\right] \\ \mu_4 &= b^4 E\left[\sum_{j=0}^{4} \binom{4}{j} (-1)^j \, (Y)^{4-j} \, (\nu_1)^j\right] \\ &= b^4 E\left[Y^4 - 4Y^3\nu_1 + 6Y^2\nu_1^2 - 4Y\nu_1^3 + \nu_1^4\right] \\ &= \frac{1}{\lambda_2^4}\left(\nu_4 - 4\nu_1\nu_3 + 6\nu_1^2\nu_2 - 3\nu_1^4\right).\end{aligned}$$

Finally, we need to determine expressions for the raw moments of Y. To this end, we have to evaluate

$$E\left[Y^k\right] = \nu_k = \int_0^1 \tilde{Q}(p)^k dp = \int_0^1 \left(\frac{p^{\lambda_3}}{\lambda_3} - \frac{(1-p)^{\lambda_4}}{\lambda_4}\right)^k dp$$

Again using binomial expansion, we can rewrite this integral as

$$\begin{aligned}
\nu_k &= \int_0^1 \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(\frac{p^{\lambda_3}}{\lambda_3}\right)^{k-j} - \left(\frac{(1-p)^{\lambda_4}}{\lambda_4}\right)^j dp \\
&= \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{\lambda_3^{k-j}\lambda_4^j} \int_0^1 \left(p^{\lambda_3(k-j)} - (1-p)^{\lambda_4 j}\right) dp \\
&= \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{\lambda_3^{k-j}\lambda_4^j} \beta\left(\lambda_3(k-j)+1, \lambda_4 j + 1\right),
\end{aligned}$$

where $\beta(\cdot,\cdot)$ is the β-function. Observe that the β-function is only well defined if all arguments are positive. This requires that

$$\lambda_3(k-j)+1 > 0 \quad \text{and} \quad \lambda_4 j + 1 > 0$$

for all k, j. This equality can only be binding if $\lambda_3, \lambda_4 < 0$. Since $j \leq k$ we can rewrite the above inequality as

$$\min(\lambda_3, \lambda_4) > -\frac{1}{k}.$$

Observe that the RHS in the above is decreasing in k. Therefore, if we target at matching moments up to $k = 4$, the constraint reads as $\min(\lambda_3, \lambda_4) > -\frac{1}{4}$.

We can also write out ν_k, for $k = 1, \dots, 4$ explicitly as functions of λ_3, λ_4 as:

$$
\begin{aligned}
\nu_1 &= \sum_{j=0}^{1} \binom{1}{j} \frac{(-1)^j}{\lambda_3^{1-j}\lambda_4^j} \beta\left(\lambda_3(1-j)+1, \lambda_4 j+1\right) \\
&= \frac{1}{\lambda_3}\beta\left(\lambda_3+1, 1\right) - \frac{1}{\lambda_4}\beta\left(1, \lambda_4+1\right) \\
&= \frac{1}{\lambda_3(\lambda_3+1)} - \frac{1}{\lambda_4(\lambda_4+1)} \\
\nu_2 &= \sum_{j=0}^{2} \binom{2}{j} \frac{(-1)^j}{\lambda_3^{2-j}\lambda_4^j} \beta\left(\lambda_3(2-j)+1, \lambda_4 j+1\right) = \nu_1(\lambda_3, \lambda_4) \\
&= \frac{1}{\lambda_3^2}\beta\left(2\lambda_3+1, 1\right) - 2\frac{1}{\lambda_3\lambda_4}\beta\left(\lambda_3+1, \lambda_4+1\right) + \frac{1}{\lambda_4^2}\beta\left(1, 2\lambda_4+1\right) \\
&= \frac{1}{\lambda_3^2\left(2\lambda_3+1\right)} + \frac{1}{\lambda_4^2\left(2\lambda_4+1\right)} - 2\frac{1}{\lambda_3\lambda_4}\beta\left(\lambda_3+1, \lambda_4+1\right) = \nu_2(\lambda_3, \lambda_4) \\
\nu_3 &= \sum_{j=0}^{3} \binom{3}{j} \frac{(-1)^j}{\lambda_3^{3-j}\lambda_4^j} \beta\left(\lambda_3(3-j)+1, \lambda_4 j+1\right) \\
&= \frac{1}{\lambda_3^3}\beta\left(3\lambda_3+1, 1\right) - \frac{3}{\lambda_3^2\lambda_4}\beta\left(2\lambda_3+1, \lambda_4+1\right) + \frac{3}{\lambda_3\lambda_4^2}\beta\left(\lambda_3+1, 2\lambda_4+1\right) - \frac{1}{\lambda_4^3}\beta\left(1, 3\lambda_4+1\right) \\
&= \frac{1}{\lambda_3^3(3\lambda_3+1)} - \frac{1}{\lambda_4^3(3\lambda_4+1)} - \frac{3}{\lambda_3^2\lambda_4}\beta\left(2\lambda_3+1, \lambda_4+1\right) + \frac{3}{\lambda_3\lambda_4^2}\beta\left(\lambda_3+1, 2\lambda_4+1\right) = \nu_3(\lambda_3, \lambda_4) \\
\nu_4 &= \sum_{j=0}^{4} \binom{4}{j} \frac{(-1)^j}{\lambda_3^{4-j}\lambda_4^j} \beta\left(\lambda_3(4-j)+1, \lambda_4 j+1\right) \\
&= \frac{1}{\lambda_3^4}\beta\left(4\lambda_3+1, 1\right) - \frac{4}{\lambda_3^3\lambda_4}\beta\left(3\lambda_3+1, 2\lambda_4+1\right) + \frac{6}{\lambda_3^2\lambda_4^2}\beta\left(2\lambda_3+1, 2\lambda_4+1\right) - \frac{4}{\lambda_3\lambda_4^3}\beta\left(\lambda_3+1, 3\lambda_4+1\right) + \\
&\quad \frac{1}{\lambda_4^4}\beta\left(1, 4\lambda_4+1\right) \\
&= \frac{1}{\lambda_3^4(4\lambda_3+1)} + \frac{1}{\lambda_4^4(4\lambda_4+1)} - \frac{4}{\lambda_3^3\lambda_4}\beta\left(3\lambda_3+1, 2\lambda_4+1\right) - \frac{4}{\lambda_3\lambda_4^3}\beta\left(\lambda_3+1, 3\lambda_4+1\right) + \\
&\quad \frac{6}{\lambda_3^2\lambda_4^2}\beta\left(2\lambda_3+1, 2\lambda_4+1\right) = \nu_4(\lambda_3, \lambda_4).
\end{aligned}
$$

From the above observe that the third and fourth central moments μ_3, μ_4 of random variable X are only functions of λ_3, λ_4. Therefore, the procedure is to determine λ_3, λ_4 jointly to target μ_3, μ_4 under the parameter restriction $\min(\lambda_3, \lambda_4) > -\frac{1}{4}$. Next, we can successively determine λ_2 from targeting μ_2 and, finally, λ_1 by targeting μ_1.

S.B A Numerical Example of the Two-Period Model

In this supplementary appendix, we present a quantitative illustration of the two-period model in order to show that higher-order income risk (in logs) may indeed lead to lower precautionary savings and utility gains. Specifically, we consider three different parameterizations of discrete PDFs $\Psi(\varepsilon)$ based on Proposition S.B.1: NORM is a symmetric distribution with a kurtosis of $\alpha_4 = 3$ as for a normal distribution. Distribution LK is also symmetric but strongly leptokurtic with a kurtosis of $\alpha_4 = 30$, and distribution LKSW additionally introduces left-skewness of $\alpha_3 = -5$. For all distributions we set the variance $\mu_2^\varepsilon = 0.5$. Throughout we normalize such that $\mathbb{E}[\exp(\varepsilon)] = 1$. To investigate the role of higher-order risk attitudes we consider two parametrizations with $\theta \in \{1, 4\}$. Throughout, we set the IES γ equal to 1, thus we focus on risk sensitive preferences.

S.B.1 Shocks

The shock ε in this two-period model is taken to be discrete. Specifically, we consider a simple lottery such that $\varepsilon \in \{\varepsilon_l, \varepsilon_0, \varepsilon_h\}$ with $\varepsilon_l < \varepsilon_0 < \varepsilon_h$ and respective probabilities $\{(1-p) \cdot q, p, (1-p) \cdot (1-q)\}$. This simple structure enables us to derive a parametrization with a closed form representation for the variance, skewness and kurtosis of the shock process, as stated in the following proposition:[2]

Proposition S.B.0 Let $\varepsilon \in \{\varepsilon_l, \varepsilon_0, \varepsilon_h\}$, drawn with respective probabilities $\{(1-p) \cdot q, p, (1-p) \cdot (1-q)\}$. Then, if and only if $\alpha_4 > 1$ and, for $\alpha_3 \neq 0$ in addition

1. either $\alpha_3 \in (0, \sqrt{\alpha_4 - 1})$
2. or $\alpha_3 \in (-\sqrt{\alpha_4 - 1}, 0)$,

[2] Our approach extends **?**), who analyzes skewness using a two-point distribution, to the fourth moment.

we match $\mu_2, \alpha_3, \alpha_4$, with the normalization $E[\exp(\varepsilon)] = 1$ by choosing

$$q = \frac{1}{2} \begin{cases} +\frac{1}{2}\sqrt{1 - \frac{4\frac{\alpha_4}{\alpha_3^2}-4}{4\frac{\alpha_4}{\alpha_3^2}-3}} & \text{if } \alpha_3 > 0 \\ -\frac{1}{2}\sqrt{1 - \frac{4\frac{\alpha_4}{\alpha_3^2}-4}{4\frac{\alpha_4}{\alpha_3^2}-3}} & \text{if } \alpha_3 < 0 \\ 0.5 & \text{if } \alpha_3 = 0 \end{cases}$$

$$p = \begin{cases} 1 - \frac{(2q-1)^2}{q(1-q)\alpha_3^2} & \text{if } \alpha_3 \neq 0 \\ 1 - \frac{1}{\alpha_4} & \text{if } \alpha_3 = 0 \end{cases}$$

$$\Delta_\varepsilon = \begin{cases} \frac{\sqrt{\mu_2}\alpha_3}{2q-1} & \text{if } \alpha_3 \neq 0 \\ 2\sqrt{\mu_2}\sqrt{\alpha_4} & \text{if } \alpha_3 = 0, \end{cases}$$

and

$$\begin{aligned} \varepsilon_l &= -\ln\left[p\exp\left((1-q)\Delta_\varepsilon\right) + (1-p)\left(q + (1-q)\exp(\Delta_\varepsilon)\right)\right] \\ \varepsilon_0 &= \varepsilon_l + (1-q)\Delta_\varepsilon \\ \varepsilon_h &= \varepsilon_l + \Delta_\varepsilon. \end{aligned}$$

Proof. See Section S.B.5. □

This representation of risk is useful because it enables us to transparently illustrate how higher-order income risk affects the distribution using a very simple structure with a closed-form solution from payoffs to the respective moments of higher-order income risk.

The upper part of Table S.B.1 summarizes the moments for the calibration of ε for these three distributions. The lower part shows how this translates into respective moments in level of the innovation, $\exp(\varepsilon)$. Going from distribution NORM to distribution LK we observe that not only the kurtosis increases strongly but also the variance. Simultaneously, the distribution becomes more skewed to the right. Thus, whether the higher kurtosis of the innovation ε also leads to welfare losses (or a strong increase in precautionary savings) depends on whether the effects on the variance and kurtosis dominate those on the skewness, cf. equations (1) and (2).

In turn, going from distribution NORM to distribution LKSW we observe that the distribution is now more skewed to the left and features a higher kurtosis. However, at the same time, the variance goes down quite strongly. Thus, whether the simultaneously higher kurtosis and lower skewness (or: increased left-skewness) of the innovation ε relative to distribution NORM lead to welfare losses (or a strong increase in precautionary savings) depends on whether the effects on the skewness and kurtosis dominate those on the variance, again see equations (1) and (2).

Table S.B.1: 2-Period Model: Shocks, standardized moments

Moments of Innovation in Logs, ε			
	μ_2^ε	α_3^ε	α_4^ε
NORM	0.5	0	3
LK	0.5	0	30
LKSW	0.5	-5	30
Moments of Innovation in Levels, $\exp(\varepsilon)$			
	$\mu_2^{\exp(\varepsilon)}$	$\alpha_3^{\exp(\varepsilon)}$	$\alpha_4^{\exp(\varepsilon)}$
NORM	0.5868	1.4885	3.7882
LK	11.6316	7.5458	57.9669
LKSW	0.1039	0.5684	4.8371

Notes: Standardized moments of the discrete shock distribution.

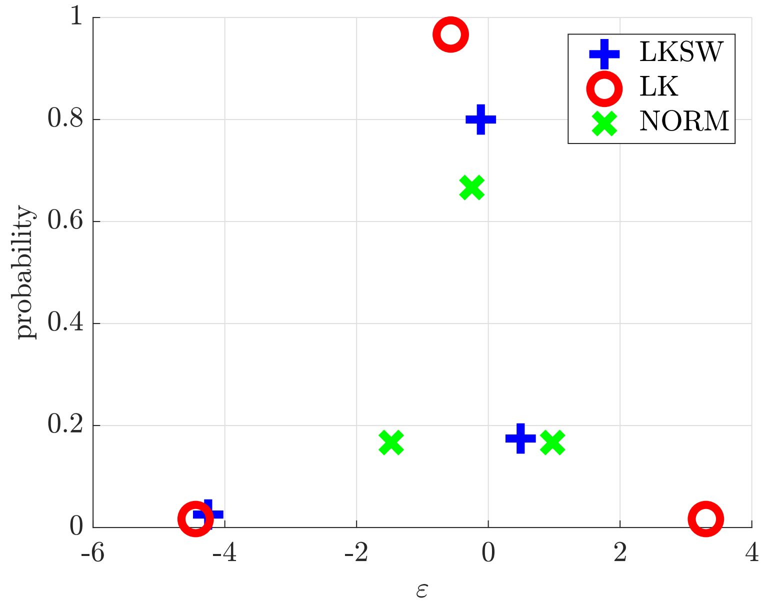
Table S.B.2: 2-Period Model: Shocks, central moments

Moments of Innovation in Logs, ε			
	μ_2^ε	μ_3^ε	μ_4^ε
NORM	0.5	0	0.75
LK	0.5	0	7.5
LKSW	0.5	-1.7678	7.5
Moments of Innovation in Levels, $\exp(\varepsilon)$			
	$\mu_2^{\exp(\varepsilon)}$	$\mu_3^{\exp(\varepsilon)}$	$\mu_4^{\exp(\varepsilon)}$
NORM	0.5868	0.6691	1.3045
LK	11.6316	299.3406	7842.5727
LKSW	0.1039	0.0190	0.0523

Notes: Central moments of the discrete shock distribution.

Figure S.B.1 plots the the corresponding PDFs $\Psi(\varepsilon)$. Relative to NORM, the distribution LK leads to a fanning out of the shocks. As can be seen for the realization of $\exp(\varepsilon_0)$ this induces a shift of the shock realizations to the left such that $\mathbb{E}[\varepsilon]$ is reduced from -0.24 to -0.57. Moving from distribution LK to distribution LKSW by additionally introducing skewness shifts the probability mass to the left tail such that $\mathbb{E}[\varepsilon]$ increases to -0.11. From this observation we know from Proposition 1 that with logarithmic utility ($\theta = 1$), we have welfare losses from the symmetric and leptokurtic distribution LK and welfare gains for the additionally left skewed distribution LKSW if households do not have access to a savings technology.

Figure S.B.1: Distribution of ε



Notes: Distribution function of the discrete shock with three points as in Proposition S.B.1 under the three scenarios NORM, LK, and LKSW.

S.B.2 Allocations

Table S.B.3 reports results on allocations, assuming that households have access to a savings technology. Increasing risk attitude coefficient θ leads to more precautionary savings and reduces the differences in precautionary savings across scenarios. Holding θ constant, compared to the distribution NORM we observe more precautionary savings for distribution LK and thus the effects of increased variance and kurtosis dominate the effects of higher skewness. In contrast, with θ constant we observe less precautionary savings for distribution LKSW and thus the effects of the lower variance dominate the effects of higher kurtosis and left-skewness.

Table S.B.4 displays the welfare consequence if there is no access to a savings technology under a binding budget constraint in column NST and with access in column ST. First, with $\theta = 1$, the distribution LKSW leads to utility gains. Thus, for our shock parametrization, the positive welfare effects of lower skewness dominate the losses of an increased kurtosis. This is true for both scenarios NST, cf. Proposition 1, as well as for scenario ST. Second, under NST utility consequences are strongly increasing in θ, as we learned from equation (1). Third, both gains and losses decrease in scenario ST compared to scenario NST. The rea-

Table S.B.3: Results from 2-Period Model: Allocations

	c_0	$\mathbb{E}[c_1]$	a_1
Risk Aversion, $\theta = 1$			
NORM	0.837	1.162	0.162
LK	0.773	1.226	0.226
LKSW	0.895	1.104	0.104
Risk Aversion, $\theta = 4$			
NORM	0.671	1.328	0.328
LK	0.662	1.337	0.337
LKSW	0.614	1.385	0.385

Notes: Allocations in the two-period model.

son is the precautionary savings response, which reduces utility losses from risk in both the denominator and the numerator of the CEV calculation. Fourth, as a consequence of the precautionary savings response, absolute values of the CEV are lower with higher risk aversion in scenario ST. This shows that the utility consequences of higher-order risk, expressed in terms of CEVs, may be non-monotonic in the degree of risk aversion.

Table S.B.4: Results from 2-Period Model: CEV

	NST	ST
Risk Aversion, $\theta = 1$		
LK	-14.82%	-11.75%
LKSW	7.03%	6.76%
Risk Aversion, $\theta = 4$		
LK	-66.20%	-3.22%
LKSW	-65.35%	5.66%

Notes: CEV relative to NORM. NST: no access to savings technology. ST: assess to savings technology.

S.B.3 Decomposition of Consumption Equivalent Variations

Table S.B.5 reports the results for the decomposition of the CEV, for sake of brevity only for $\theta = 1$ and with access to a savings technology (ST). With this calibration, most of the changes appear in the cross-sectional distribution effect.

Table S.B.5: Results from 2-Period Model: Decomposition of CEV for Log Utility

CEV	g_c	g_c^{mean}	g_c^{lcd}	g_c^{csd}
Baseline				
LK	-11.75%	0	-2.35%	-9.40%
LKSW	6.76%	0	2.16%	4.59%
Impatience				
LK	-11.04%	0	-9.82%	-1.22%
LKSW	-4.10%	0	-10.50%	6.40%
Positive Interest Rate				
LK	-5.56%	2.65%	-4.92%	-3.30%
LKSW	1.70%	2.63%	-4.85%	3.93%
Borrowing Constraint				
LK	-5.04%	0.34%	-0.65%	-4.73%
LKSW	2.26%	0.13%	-0.26%	2.38%

Notes: CEV relative to NORM for $\theta = 1, \rho = 1$ for scenario ST. LK: leptokurtik distribution, LKSW: leptokurtik and skewed distribution.

S.B.4 Additional Model Elements

For the remaining exercises we add step by step model elements included in the quantitative model. Throughout, we take $\theta = \frac{1}{\rho} = 1$ and only analyze the welfare consequences in terms of the consumption equivalent variation. Results are contained in the remaining rows of Table S.B.5 .

Impatience. We first add a period discount factor β of 0.96, such that the discount factor accounting for the 40-year periodicity is $0.96^{40} \approx 0.19$. This introduces a life-cycle savings motive into the model and preferences now write as (for $\rho \neq 1$)

$$U = \frac{1}{1-\rho}\left((1-\tilde{\beta})c_0^{1-\frac{1}{\rho}} + \tilde{\beta} v\left(c_1, \theta, \Psi\right)^{1-\frac{1}{\rho}}\right),$$

where $\tilde{\beta} = \frac{\beta}{1+\beta}$ and β is the raw time discount factor. As a consequence of discounting, the life-cycle distribution effect becomes more potent. Households now take on debt to finance consumption when young. Given the riskiness of second period consumption, borrowing is much lower in distributions LK and LKSW than in distribution NORM. Therefore, the life-cycle distribution effect is strongly negative.

Positive Returns. Next, we also assume a positive interest rate on savings with an annual raw interest rate of 2%. Given the length of each model period of 40 real life years, this corresponds to $R = 1.02^{40} \approx 2.2$. Thus, the budget constraints now write as

$$a_1 = y_0 - c_0, \qquad c_1 \leq a_1 \cdot R + y_1.$$

Table S.B.5 shows that now the mean effect is non-zero. The reason is that savings are inter-temporally shifted at a non-zero rate so that average consumption increases. Results also show that the aforementioned life-cycle effects are muted. Still the life-cycle distribution effects are negative.

Borrowing Constraints. Next, we add occasionally binding borrowing constraints at zero borrowing, i.e., we add the constraint

$$a_1 \geq 0.$$

For the chosen parametrization this constraint turns out to be binding only in scenario NORM. Since households are thus worse off in NORM relative to the other scenarios, welfare losses in distribution LK decrease and gains in distribution LKSW increase.

Throughout all these scenarios, we observe that the cross-sectional distribution effect is negative in scenario LK, and positive in scenario LKSW.

S.B.5 Proof of Proposition S.B.1

Proof. Take $\varepsilon_0 = \mu_1$, thus

$$\begin{aligned} \mu_1 &= p\varepsilon_0 + (1-p)\left(q\varepsilon_l + (1-q)\varepsilon_h\right) \\ &= p\mu_1 + (1-p)\left(q\varepsilon_l + (1-q)\varepsilon_h\right) \\ \Leftrightarrow \quad \mu_1 &= q\varepsilon_l + (1-q)\varepsilon_h. \end{aligned}$$

Now, let $\varepsilon_h = \varepsilon_l + \Delta_\varepsilon$ to get

$$\begin{aligned} \mu_1 &= q\varepsilon_l + (1-q)\left(\varepsilon_l + \Delta_\varepsilon\right) \\ &= \varepsilon_l + (1-q)\Delta_\varepsilon. \end{aligned}$$

For the variance we get

$$\begin{aligned}
\mu_2 &= (1-p)\left(q(\varepsilon_l-\mu_1)^2+(1-q)(\varepsilon_h-\mu_1)^2\right)\\
&= (1-p)\left(q\left(\varepsilon_l-(\varepsilon_l+(1-q)\Delta_\varepsilon)\right)^2+(1-q)\left(\varepsilon_h-(\varepsilon_l+(1-q)\Delta_\varepsilon)\right)^2\right)\\
&= (1-p)\left(q(1-q)^2+(1-q)q^2\right)\Delta_\varepsilon^2\\
&= (1-p)q(1-q)\Delta_\varepsilon^2.
\end{aligned}$$

For the third central moment μ_3 we get

$$\begin{aligned}
\mu_3 &= (1-p)\left(q(\varepsilon_l-\mu_1)^3+(1-q)(\varepsilon_h-\mu_1)^3\right)\\
&= (1-p)\left(q\left(\varepsilon_l-(\varepsilon_l+(1-q)\Delta_\varepsilon)\right)^3+(1-q)\left(\varepsilon_h-(\varepsilon_l+(1-q)\Delta_\varepsilon)\right)^3\right)\\
&= (1-p)\left(-q(1-q)^3+(1-q)q^3\right)\Delta_\varepsilon^3\\
&= (1-p)q(1-q)\left(-(1-q)^2+q^2\right)\Delta_\varepsilon^3\\
&= (1-p)q(1-q)(2q-1)\Delta_\varepsilon^3
\end{aligned}$$

and we can thus write the skewness α_3 as

$$\alpha_3 = \frac{\mu_3}{\sqrt{\mu_2}^3} = \frac{2q-1}{\sqrt{(1-p)q(1-q)}}.$$

For the fourth central moment μ_4 we get

$$\begin{aligned}
\mu_4 &= (1-p)\left(q(\varepsilon_l-\mu_1)^4+(1-q)(\varepsilon_h-\mu_1)^4\right)\\
&= (1-p)\left(q\left(\varepsilon_l-(\varepsilon_l+(1-q)\Delta_\varepsilon)\right)^4+(1-q)\left(\varepsilon_h-(\varepsilon_l+(1-q)\Delta_\varepsilon)\right)^4\right)\\
&= (1-p)\left(q(1-q)^4+(1-q)q^4\right)\Delta_\varepsilon^4\\
&= (1-p)q(1-q)\left((1-q)^3+q^3\right)\Delta_\varepsilon^4\\
&= (1-p)q(1-q)\left((1-2q+q^2)(1-q)+q^3\right)\Delta_\varepsilon^4\\
&= (1-p)q(1-q)\left(1-3q+3q^2\right)\Delta_\varepsilon^4
\end{aligned}$$

and can therefore write the kurtosis as

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{3q^2-3q+1}{(1-p)q(1-q)}.$$

To summarize, the terms we seek to match are

$$\mu_2 = (1-p)q(1-q)\Delta_\varepsilon^2, \tag{S.B.6a}$$

$$\alpha_3 = \frac{2q-1}{\sqrt{(1-p)q(1-q)}}, \tag{S.B.6b}$$

$$\alpha_4 = \frac{3q^2-3q+1}{(1-p)q(1-q)}. \tag{S.B.6c}$$

To obtain $\alpha_4 > 0$ we require $p \in (0,1)$, $q \in (0,1)$ and

$$\begin{aligned} & q^2 - q + \frac{1}{3} > 0 \\ \Leftrightarrow \qquad & \left(q - \frac{1}{2}\right)^2 > -\frac{1}{12} \end{aligned}$$

which always holds.

Let us next characterize the solution according to the following case distinction:

1. $\alpha_3 = 0$. Then we obviously have $q = 1 - q = 0.5$. We can accordingly rewrite (S.B.6a) and (S.B.6c) as

$$\begin{aligned} \mu_2 &= (1-p)\frac{1}{4}\Delta_\varepsilon^2, \\ \alpha_4 &= \frac{1}{(1-p)}, \end{aligned}$$

   and therefore

$$\begin{aligned} q &= \frac{1}{2} \\ p &= 1 - \frac{1}{\alpha_4} \\ \Delta_\varepsilon &= 2\sqrt{\mu_2}\sqrt{\alpha_4} \end{aligned}$$

   characterizes the solution. Notice that $\alpha_4 > 0$ and thus $p < 1$. To get $p > 0$ we require

$$1 - \frac{1}{\alpha_4} > 0 \qquad \Leftrightarrow \qquad \alpha_4 > 1.$$

2. $\alpha_3 \neq 0$. From (S.B.6a) we get

$$(1-p)q(1-q) = \frac{\mu_2}{\Delta_\varepsilon^2}$$

Using this in (S.B.6b) and (S.B.6c) we get

$$\alpha_3 = \frac{(2q-1)\Delta_\varepsilon}{\sqrt{\mu_2}}, \tag{S.B.7a}$$

$$\alpha_4 = \frac{(3q^2-3q+1)\Delta_\varepsilon^2}{\mu_2}. \tag{S.B.7b}$$

Now use (S.B.7a) in (S.B.7b) to get

$$\begin{aligned} & \frac{(3q^2-3q+1)}{(2q-1)^2} = \frac{\alpha_4}{\alpha_3^2} \\ \Leftrightarrow \quad & (3q^2-3q+1) = \frac{\alpha_4}{\alpha_3^2}\left(4q^2-4q+1\right) \\ \Leftrightarrow \quad & q^2\left(4\frac{\alpha_4}{\alpha_3^2}-3\right) - q\left(4\frac{\alpha_4}{\alpha_3^2}-3\right) + \frac{\alpha_4}{\alpha_3^2} - 1 = 0 \\ \Leftrightarrow \quad & q^2 - q + \frac{\frac{\alpha_4}{\alpha_3^2}-1}{4\frac{\alpha_4}{\alpha_3^2}-3} = 0 \end{aligned}$$

and thus

$$q_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{\underbrace{1 - \frac{4\frac{\alpha_4}{\alpha_3^2}-4}{4\frac{\alpha_4}{\alpha_3^2}-3}}_{=\Psi}} \tag{S.B.8}$$

Thus, the first restriction for $q_\pm \in (0,1)$ is that $\Psi > 0$. Consider the following case distinction:

(a) $4\frac{\alpha_4}{\alpha_3^2} - 3 > 0 \;\Leftrightarrow\; \frac{\alpha_4}{\alpha_3^2} > \frac{3}{4}$: Then

$$\begin{aligned} & 1 - \frac{4\frac{\alpha_4}{\alpha_3^2}-4}{4\frac{\alpha_4}{\alpha_3^2}-3} > 0 \\ \Leftrightarrow \quad & 4\frac{\alpha_4}{\alpha_3^2} - 3 > 4\frac{\alpha_4}{\alpha_3^2} - 4 \\ \Leftrightarrow \quad & 4 > 3 \end{aligned}$$

and thus for $\frac{\alpha_4}{\alpha_3^2} > \frac{3}{4}$ we get $\Psi > 0$.

(b) $4\frac{\alpha_4}{\alpha_3^2} - 3 < 0 \;\Leftrightarrow\; \frac{\alpha_4}{\alpha_3^2} < \frac{3}{4}$ then we obviously get a contradiction.

Thus, we require $\alpha_4 > \frac{3}{4}\alpha_3^2$.

Next, for both the positive and the negative root, we further require $\Psi < 1$. Again investigate the case $\alpha_4 > \frac{3}{4}\alpha_3^2$. We get

$$\begin{aligned}
& 1 - \frac{4\frac{\alpha_4}{\alpha_3^2} - 4}{4\frac{\alpha_4}{\alpha_3^2} - 3} < 1 \\
\Leftrightarrow \quad & \frac{4\frac{\alpha_4}{\alpha_3^2} - 4}{4\frac{\alpha_4}{\alpha_3^2} - 3} > 0 \\
\Leftrightarrow \quad & 4\frac{\alpha_4}{\alpha_3^2} - 4 > 0 \\
\Leftrightarrow \quad & \frac{\alpha_4}{\alpha_3^2} > 1
\end{aligned}$$

and thus a necessary and sufficient condition for $q_\pm \in (0, 1)$ is:

$$\alpha_4 > \alpha_3^2. \tag{S.B.9}$$

Since $\alpha_3 = \frac{(2q-1)\Delta_\varepsilon}{\sqrt{\mu_2}}$ and since $\Delta_\varepsilon > 0$ (by construction) and $\sqrt{\mu_2} > 0$ we choose the positive root $q^\star = q_+$ for a right-skewed distribution with $\alpha_3 > 0$ and the negative root $q^\star = q_-$ to model a left-skewed with $\alpha_3 < 0$.

We next get from (S.B.7a) that

$$\Delta_\varepsilon = \frac{\sqrt{\mu_2}\alpha_3}{2q^\star - 1}$$

and from (S.B.6a) that

$$p = 1 - \frac{\mu_2}{q^\star(1-q^\star)\Delta_\varepsilon^2} = 1 - \frac{(2q^\star - 1)^2}{q^\star(1-q^\star)\alpha_3^2}. \tag{S.B.10}$$

We have already established that under condition (S.B.9) $q^\star \in (0, 1)$. Next, we need to establish conditions such that $p \in (0, 1)$. From (S.B.10) we observe that $q^\star \in (0, 1)$ gives $p < 1$. Also observe that $p > 0$ is equivalent to

$$\alpha_3^2 > \frac{(2q^\star - 1)^2}{q^\star(1-q^\star)} \tag{S.B.11}$$

(a) Case $\alpha_3 < 0$: Recall that for this case we take the negative root $q_-^\star$, where

$$q_-^\star = \frac{1}{2} - \frac{1}{2}\sqrt{\Psi} > 0.$$

for $\Psi \in (0, 1)$ iff $\alpha_4 > \alpha_3^2$. Thus the case $\alpha_3 < 0$ implies that $\alpha_3 > -\sqrt{\alpha_4}$. Next observe that

$$(2q^\star - 1)^2 = (1 - \sqrt{\Psi} - 1)^2 = \Psi$$

and

$$\begin{aligned} q^\star(1-q^\star) &= \left(\frac{1}{2} - \frac{1}{2}\sqrt{\Psi}\right)\left(\frac{1}{2} + \frac{1}{2}\sqrt{\Psi}\right) \\ &= \frac{1}{4} - \frac{1}{4}\Psi = \frac{1}{4}\left(1 - \Psi\right). \end{aligned}$$

Thus condition (S.B.11) can be rewritten as

$$\begin{aligned} & \alpha_3^2 > \frac{(2q^\star - 1)^2}{q^\star(1-q^\star)} = \frac{4\Psi}{1-\Psi} \\ \Leftrightarrow \quad & \alpha_3^2(1-\Psi) > 4\Psi \\ \Leftrightarrow \quad & \alpha_3^2 \frac{\frac{4\alpha_4}{\alpha_3^2} - 4}{4\frac{\alpha_4}{\alpha_3^2} - 3} > 4\left(1 - \frac{\frac{4\alpha_4}{\alpha_3^2} - 4}{4\frac{\alpha_4}{\alpha_3^2} - 3}\right) \\ \Leftrightarrow \quad & \alpha_3^2\left(\frac{\alpha_4}{\alpha_3^2} - 1\right) > 4\frac{\alpha_4}{\alpha_3^2} - 3 - \left(4\frac{\alpha_4}{\alpha_3^2} - 4\right) \\ \Leftrightarrow \quad & \alpha_4 - \alpha_3^2 > 1 \\ \Leftrightarrow \quad & \alpha_3 > -\sqrt{\alpha_4 - 1}, \text{ since } \alpha_3 < 0 \end{aligned}$$

which also implies that we require $\alpha_4 > 1$. Since $-\sqrt{\alpha_4 - 1} > -\sqrt{\alpha_4}$ we thus obtain as a necessary and sufficient condition for the case $\alpha_3 < 0$

$$\alpha_4 > 1 \text{ and } \alpha_3 > -\sqrt{\alpha_4 - 1} \tag{S.B.12}$$

to get $q \in (0, \frac{1}{2})$, $p \in (0,1)$ and $\Delta\epsilon > 0$.

(b) Case $\alpha_3 > 0$: Recall that for this case we take the positive root $q_+^\star$ where

$$q_+^\star = \frac{1}{2} + \frac{1}{2}\sqrt{\Psi} > 0.$$

for $\Psi \in (0,1)$ iff $\alpha_4 > \alpha_3^2$ and thus $\alpha_3 < \sqrt{\alpha_4}$. Thus

$$(2q^\star - 1)^2 = \Psi$$

and

$$q^\star(1-q^\star) = \left(\frac{1}{2}+\frac{1}{2}\sqrt{\Psi}\right)\left(\frac{1}{2}-\frac{1}{2}\sqrt{\Psi}\right)$$
$$= \frac{1}{4}-\frac{1}{4}\Psi = \frac{1}{4}\left(1-\Psi\right).$$

and following the steps above we thus get

$$\alpha_4 - \alpha_3^2 > 1$$
$$\Leftrightarrow \qquad \alpha_3 < \sqrt{\alpha_4 - 1},$$

Since $\sqrt{\alpha_4 - 1} < \sqrt{\alpha_4}$ we thus obtain as a necessary and sufficient condition for the case $\alpha_3 > 0$

$$\alpha_4 > 1 \;\text{ and }\; \alpha_3 < \sqrt{\alpha_4 - 1} \tag{S.B.13}$$

to get $q \in (\frac{1}{2}, 1)$, $p \in (0,1)$ and $\Delta\epsilon > 0$.

Finally, for ε_l given, the mean of the exponent of the random variable x is given by

$$E\left[\exp(x)\right] = p\exp\left(\varepsilon_l + (1-q)\Delta_\varepsilon\right) + (1-p)\left(q\exp\left(\varepsilon_l\right) + (1-q)\exp\left(\varepsilon_l + \Delta_\varepsilon\right)\right)$$
$$= \exp(\varepsilon_l)\left[p\exp\left((1-q)\Delta_\varepsilon\right) + (1-p)\left(q + (1-q)\exp\left(\Delta_\varepsilon\right)\right)\right].$$

Normalizing $E\left[\exp(x)\right] = 1$ we thus get

$$\varepsilon_l = -\ln\left[p\exp\left((1-q)\Delta_\varepsilon\right) + (1-p)\left(q + (1-q)\exp\left(\Delta_\varepsilon\right)\right)\right].$$

□