Online Appendix

Optimal Taxes on Capital in the OLG Model with Uninsurable Idiosyncratic Income Risk

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A Derivation of the Current Generations $GE(s)$ Effects

From equations (13a) and (13b) we find that

$$w'(s) = (1-\alpha)\alpha\left[k'(s)\right]^{\alpha-1}\frac{dk'(s)}{ds} = (1-\alpha)\alpha\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha}\left[s\right]^{\alpha-1}$$

$$R'(s) = \alpha(\alpha-1)\left[k'(s)\right]^{\alpha-2}\frac{dk'(s)}{ds} = \alpha(\alpha-1)\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha-1}\left[s\right]^{\alpha-2}$$

and thus

$$\kappa\eta w'(s) + (1-\kappa)(1-\alpha)k^{\alpha}R'(s)s = (1-\alpha)\alpha\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha}\left[s\right]^{\alpha-1}\left[\kappa\eta-1\right].$$

For a general period utility function $u(.)$ the general equilibrium effect reads as

$$GE(s) = (1-\alpha)\alpha\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha}\left[s\right]^{\alpha-1}\beta\int u'\left(c^{o}(\eta)\right)\left[\kappa\eta-1\right]d\Psi(\eta). \quad (1)$$

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If the utility function is logarithmic, equation (1) specializes, after substitution for $c^o(\eta)$ from the budget constraint, to equation $(18b)$ in the main text. Note that

$$\begin{aligned}\int u'\left(c^o(\eta)\right)[\kappa\eta-1]\,d\Psi(\eta) &= (\kappa-1)\int u'(c^o(\eta))d\Psi(\eta)+Cov\left[u'(c^o(\eta)),(\kappa\eta-1)\right]\\ &< (\kappa-1)\int u'(c^o(\eta))d\Psi(\eta)<0.\end{aligned}$$

Thus, the general equilibrium effect is unambiguously negative as asserted in the main text.

B Derivation of Optimal Saving Rate for Log-Utility

B.1 Sequential Formulation

In this section we provide a full solution to the Ramsey optimal taxation problem for the case of logarithmic utility in its sequential formulation, for an arbitrary set of social welfare weights. We first recognize from the aggregate law of motion that

$$\begin{aligned}\ln(k_{t+1}) &= \ln(1-\alpha)+\ln(1-\kappa)+\alpha\ln(k_t)+\ln(s_t)\\ &= \varkappa+\sum_{i=0}^{t}\alpha^i\ln(s_{t-i})+\alpha^{t+1}\ln(k_0)\\ &= \varkappa_{t+1}+\sum_{i=0}^{t}\alpha^i\ln(s_{t-i}),\end{aligned}$$

where $\varkappa_{t+1}=\ln(1-\alpha)+\ln(1-\kappa)+\alpha^{t+1}\ln(k_0)$. Therefore the objective of the Ramsey government is given by (suppressing maximization-irrelevant constants)

$$\begin{aligned}\sum_{t=0}^{\infty}\omega_t V(k_t,s_t) &= \sum_{t=0}^{\infty}\omega_t\left[\ln(1-s_t)+\alpha\beta\ln(s_t)+\alpha\left(1+\alpha\beta\right)\ln(k_t)\right]\\ &= \chi+\sum_{t=0}^{\infty}\omega_t\left[\ln(1-s_t)+\alpha\beta\ln(s_t)+\alpha\left(1+\alpha\beta\right)\sum_{i=1}^{\infty}\alpha^{i-1}\ln(s_{t-i})\right]\\ &= \chi+\sum_{t=0}^{\infty}\left[\omega_t\ln(1-s_t)+\ln(s_t)\left(\alpha\beta\omega_t+\alpha\left(1+\alpha\beta\right)\sum_{i=t+1}^{\infty}\omega_i\alpha^{i-(t+1)}\right)\right]\end{aligned}$$

and thus the social welfare function can be expressed purely in terms of saving rates as

$$W(\{s_t\}_{t=0}^{\infty}) = \chi + \sum_{t=0}^{\infty} \omega_t \left[\ln(1-s_t) + \ln(s_t) \left(\alpha\beta + \alpha\left(1+\alpha\beta\right) \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} \right) \right],$$

where χ is a constant that depends positively on the initial capital stock k_0, but is again irrelevant for maximization. Taking first order conditions with respect to s_t and setting it to zero delivers the optimal saving rate in the main text:

$$s_t = \frac{1}{1 + \left(\alpha\beta + \alpha\left(1+\alpha\beta\right) \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} \right)^{-1}}.$$

B.2 Recursive Formulation

To obtain the closed form solution of the recursive version of the problem for $\frac{\omega_{t+1}}{\omega_t} = \theta$ by the method of undetermined coefficients guess that the value function takes the following log-linear form:

$$W(k) = \Theta_0 + \Theta_1 \ln(k).$$

Using this guess and equations $(13a)$-$(13c)$ (and writing $k_{t+1}(s_t)$ recursively as $k'(s)$) rewrite the Bellman equation (16) as:

$$\begin{aligned} W(k) &= \Theta_0 + \Theta_1 \ln(k) \qquad (2) \\ &= \max_{s\in[0,1]} \Big\{ \ln((1-s)(1-\kappa)\left(1-\alpha\right)k^{\alpha}) \\ &\quad + \beta \int \ln\left(\kappa\eta w(s) + R(s)k'(s)\right) d\Psi(\eta) + \theta W(k') \Big\} \\ &= \max_{s\in[0,1]} \Big\{ \ln((1-s)(1-\kappa)\left(1-\alpha\right)k^{\alpha}) \\ &\quad + \beta \int \ln\left([\kappa(1-\alpha)\eta + \alpha][s(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha}\right) d\Psi(\eta) + \theta W(k') \Big\} \\ &= (1+\alpha\beta)\ln((1-\kappa)\left(1-\alpha\right)) + \beta \int \ln\left(\kappa\eta(1-\alpha)+\alpha\right) d\Psi(\eta) \\ &\quad + \theta\Theta_0 + \theta\Theta_1 \ln\left[(1-\kappa)(1-\alpha)\right] + \left[\alpha + \alpha^2\beta + \alpha\theta\Theta_1\right] \ln(k) \\ &\quad + \max_{s\in[0,1]} \left\{ \ln(1-s) + \left(\alpha\beta + \theta\Theta_1\right)\ln(s) \right\}. \end{aligned}$$

For the Bellman equation to hold, the coefficient Θ_1 has to satisfy

$$\Theta_1 = \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)}.$$

We also immediately recognize that the optimal saving rate chosen by the Ramsey planner is independent of the capital stock k and determined by the first order condition

$$\frac{1}{1-s} = \frac{\alpha\beta + \theta\Theta_1}{s}$$

and thus

$$s^* = \frac{\alpha\beta + \theta\Theta_1}{1+\alpha\beta+\theta\Theta_1} = \frac{\alpha(\beta+\theta)}{1+\alpha\beta} \tag{3}$$

as given by equation (20) in the main text. Plugging in s^* and Θ_1 into the Bellman equation (2) yields a linear equation in the constant Θ_0 whose solution completes the full analytical characterization of the Ramsey optimal taxation problem.

C Details of the Empirical Analysis

C.1 Sample Selection

We use data from the Panel Study of Income Dynamics (PSID), which interviews households in the United States annually from 1968 to 1997 and every other year since then.[1] The representative core sample consists of about 2,000 households in each wave, and we use data from 1977–2012.[2] Household pre-government income is defined as labor income before taxes, which we calculate as the sum of head and spouse annual labor income. We impute taxes using Taxsim, and add 50% of the estimated payroll taxes to the sum of head and spouse labor incomes to obtain pre-government income. We deflate all nominal values with the annual CPI, and select households if the household head is between 25 and 84 years of age. The minimum of household pre-government income needs to be above a constant threshold, which is defined as the income from working 520 hours at half the minimum wage.

[1] We thank Chris Busch for helping us with the data.

[2] We do not use earlier waves because of poor coverage of income transfers before the 1977 wave.

Labor Income Share. We take our pre-government income measure to compute the ratio of labor income to total income (defined as the sum of labor income and capital income) for each household in the sample and take the average. This gives 0.792, suggesting that $\alpha \approx 0.208$.

Estimate of κ. In our model, young workers have average productivity $1-\kappa$, and old workers have average productivity κ. Thus the ratio of average earnings of old to young workers is $d = \frac{\kappa}{1-\kappa}$ and thus $\kappa = \frac{d}{1+d}$. We define young workers as workers in the age range 25 to 54 and old workers as workers of age 55 to 84. As ratio of their earnings we obtain $d = 0.453$ and thus $\kappa \approx 0.312$.

Lower bound support of η. Based on our income measure we compute the ratio of the lowest income in our sample of old workers of age 55 to 84 to the median income in that group giving 3.35%.

Residual Income Variance We run a panel regression, with log income as dependent variable and time dummies, a cubic in age, a control for the number of adult household members and an additional cubic in years of education for college workers as independent variables giving a variance of 0.648.

D Overaccumulation of Capital in the Competitive Equilibrium and Positive Capital Taxation

In this section we provide the details of the relation between the solution to the Ramsey problem in the steady state and the overaccumulation of capital (a capital stock above the golden rule capital stock) in the steady state equilibrium absent government policy, including the proof of Proposition 4 in the main text.

D.1 Definitions

First, and as usual, define the golden rule capital stock as the capital stock that maximizes aggregate (per capita) steady state consumption $C = k^{\alpha} - k$. Thus, the golden rule capital stock, saving rate and associated gross real interest rate are given by:

$$\begin{aligned} k^{GR} &= \alpha^{\frac{1}{1-\alpha}} \\ s^{GR} &= \frac{\alpha}{(1-\kappa)(1-\alpha)} \\ R^{GR} &= 1. \end{aligned}$$

A capital stock and associated saving rate is inefficiently high if it is larger than the golden rule level, and thus the associated gross real interest rate is less than 1. In this case aggregate consumption can be increased by lowering the capital stock in this case.

Now let us turn to the steady state of a competitive equilibrium. In any such steady state, the gross real interest rate is related to the steady state capital stock k through

$$R = \alpha k^{\alpha-1}.$$

From the law of motion of capital (equation (8b)) in a steady state

$$k = s(1-\kappa)(1-\alpha)k^{\alpha}$$

the steady state equilibrium interest rate R is related to the saving rate s by

$$R = \frac{\alpha}{s(1-\kappa)(1-\alpha)}.$$

The steady state equilibrium saving rate s itself is given by (see equation (11))

$$s = \frac{1}{1 + [(1-\tau)\alpha\beta\Gamma]^{-1}} = \frac{(1-\tau)\alpha\beta\Gamma}{1 + (1-\tau)\alpha\beta\Gamma}$$

which leads to a steady state relation between the real interest rate and the tax rate:

$$R = \frac{\frac{1}{(1-\tau)\beta\Gamma} + \alpha}{(1-\kappa)(1-\alpha)} = R(\tau;\Gamma).$$

A higher tax rate τ reduces the saving rate, the capital stock and increases the real interest rate. Holding τ constant the steady state interest rate is decreasing in the amount of income risk measured by Γ. The steady state interest rate in the absence of government policy

$(\tau = 0)$ is given by

$$R(\tau = 0; \Gamma) = \frac{\frac{1}{\beta\Gamma} + \alpha}{(1-\kappa)(1-\alpha)}.$$

D.2 Overaccumulation of Capital in the Competitive Equilibrium

Recall that $\bar{\Gamma} = \frac{1}{\kappa(1-\alpha)+\alpha}$. The steady state competitive equilibrium in the absence of taxes has overaccumulated capital (a capital stock above the golden rule and $R(\tau = 0; \Gamma) < 1$) if and only if

$$\begin{aligned} \frac{\frac{1}{\beta\Gamma} + \alpha}{(1-\kappa)(1-\alpha)} &< 1 \\ \Gamma^{\text{eff}} := \frac{1}{[1 - \alpha - 1/\bar{\Gamma}]\beta} &< \Gamma \end{aligned} \tag{4}$$

The constant Γ^{eff} gives the first bound used in Proposition 4.

The optimal Ramsey steady state (i.e., $\theta = 1$) tax rate (see equation (21)) is given by

$$1 - \tau = \frac{1+\beta}{(1-\alpha)\,\beta\Gamma}$$

and thus the optimal Ramsey tax rate is positive, $\tau > 0$, if and only if

$$\begin{aligned} \frac{1+\beta}{(1-\alpha)\,\beta\Gamma} &< 1 \\ \Gamma^{\tau=0} : \frac{1+\beta}{(1-\alpha)\,\beta} &< \Gamma. \end{aligned} \tag{5}$$

In the proposition we made the assumption that $\beta < \frac{1}{(1-\alpha)\bar{\Gamma}-1} = \frac{\kappa(1-\alpha)+\alpha}{(1-\kappa)(1-\alpha)-\alpha}$ to insure that all cases of the proposition can occur. Under this assumption $\Gamma^{\tau=0} < \Gamma^{\text{eff}}$ and the interval in the second part of the proposition is nonempty (the equilibrium capital stock can be below the golden rule yet capital is taxed at a positive rate) since

$$\begin{aligned}
\Gamma^{\tau=0} := \frac{1+\beta}{(1-\alpha)\,\beta} &< \frac{1}{[1-\alpha-1/\bar{\Gamma}]\beta} := \Gamma^{\text{eff}} \\
\frac{1+\beta}{1-\alpha} &< \frac{1}{1-\alpha-1/\bar{\Gamma}} \\
1+\beta &< \frac{1-\alpha}{(1-\kappa)(1-\alpha)-\alpha} \\
\beta &< \frac{\kappa(1-\alpha)+\alpha}{(1-\kappa)(1-\alpha)-\alpha},
\end{aligned}$$

which holds on account of the assumption made in the proposition. Thus if the steady state competitive equilibrium capital stock is above the golden rule the optimal tax on capital is positive, but the opposite is not necessarily true. If there is no risk, however, then $\Gamma = \bar{\Gamma}$ and conditions (4) and (5) coincide:

$$\begin{aligned}
\frac{1}{[1-\alpha-1/\bar{\Gamma}]\beta} &< \bar{\Gamma} \\
\frac{1}{\beta}+1 &< (1-\alpha)\bar{\Gamma} \\
\frac{1+\beta}{(1-\alpha)\,\beta} &< \bar{\Gamma}
\end{aligned}$$

This results in the following proposition, referenced in the main text:

Proposition 1. *Let $\theta = 1$ and the assumption in Proposition 4 be satisfied. If the steady state competitive equilibrium capital stock is larger than the golden rule, the optimal Ramsey tax rate τ is positive. If η is degenerate at $\eta = 1$, then the reverse is true as well: $\tau > 0$ only if the steady state competitive equilibrium capital stock is larger than the golden rule.*

It remains to show the ranking of the savings rates in the different parts of Proposition 4. Recall that the savings rates are defined as

$$s^{CE} = \frac{1}{1+[\alpha\beta\Gamma]^{-1}} \tag{6a}$$

$$s^{*} = \frac{\alpha(1+\beta)}{1+\alpha\beta} \tag{6b}$$

$$s^{GR} = \frac{\alpha}{(1-\kappa)(1-\alpha)}. \tag{6c}$$

It then follows directly from the definition of Γ^{eff} that $s^{CE} < s^{GR}$ if and only if $\Gamma < \Gamma^{\text{eff}}$, and it follows directly from the definition of $\Gamma^{\tau=0}$ that $s^{CE} < s^*$ if and only if $\Gamma < \Gamma^{\tau=0}$. Finally, the condition on β in the proposition implies that $s^* < s^{GR}$.

E Characterization of Efficient Allocations

E.1 Characterization of Pareto Efficient Allocations

In this section we derive the solution to the unconstrained social planner problem and study whether the Ramsey government implements Pareto efficient allocations. The obvious answer is no, since an unconstrained social planner would provide full insurance against idiosyncratic η shocks, which, given the market structure, is ruled out in any competitive equilibrium. More interesting is the question how the saving rate chosen by the unconstrained planner compares to that selected by a constrained planner and the Ramsey government. The planner maximizes social welfare

$$\omega_{-1}\beta \int \ln(c_0^o(\eta_0))d\Psi(\eta_0) + \sum_{t=0}^{\infty} \omega_t \left[\ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1}))d\Psi(\eta_{t+1})\right]$$

subject just to the sequence of resource constraints

$$c_t^y + \int c_t^o(\eta_t)d\Psi(\eta_t) + k_{t+1} = k_t^\alpha.$$

We again restrict attention to geometrically declining welfare weights: $\omega_{t+1}/\omega_t = \theta \leq 1$. Trivially, the social planner provides full insurance against idiosyncratic income risk so that $c_t^o(\eta) = c_t^o$ for all η and all t. Thus the problem simplifies to

$$\max_{\{c_t^y, c_t^o, k_{t+1}\}} \omega_{-1}\beta \ln(c_0^o) + \sum_{t=0}^{\infty} \omega_t \left[\ln(c_t^y) + \beta \ln(c_{t+1}^o)\right] \qquad s.t.$$

$$c_t^y + c_t^o + k_{t+1} = k_t^\alpha$$

with $k_0 > 0$ given. The first order conditions are given by

$$\begin{aligned}
\frac{\omega_t}{c_t^y} &= \lambda_t \\
\frac{\beta\omega_{t-1}}{c_t^o} &= \lambda_t \\
\lambda_t &= \lambda_{t+1}\alpha k_{t+1}^{\alpha-1} \\
c_t^y + c_t^o + k_{t+1} &= k_t^\alpha .
\end{aligned}$$

The optimal allocation of consumption across two generations at a given time t is then given by

$$\frac{c_t^o}{c_t^y} = \frac{\beta\omega_{t-1}}{\omega_t}$$

and over time for a given generation it is characterized by

$$\frac{c_{t+1}^o}{c_t^y} = \beta\alpha k_{t+1}^{\alpha-1}.$$

In contrast to the Ramsey problem, consumption of the old in the first period is no longer irrelevant for maximization because the social planner can redistribute resources intergenerationally whereas the Ramsey planner, given the assumed restriction on instruments cannot. Thus, we characterize optimal allocations in period 0 and in an arbitrary period $t > 0$ separately.

Periods $t > 0$. Since we have assumed that $\frac{\omega_{t+1}}{\omega_t} = \theta$ we obtain

$$\frac{c_t^o}{c_t^y} = \frac{\beta\omega_{t-1}}{\omega_t} = \frac{\beta}{\theta}$$

and thus from the resource constraint we get

$$\begin{aligned}
c_t^y &= \frac{\theta}{\theta+\beta}\left(k_t^\alpha - k_{t+1}\right) \\
c_t^o &= \frac{\beta}{\theta+\beta}\left(k_t^\alpha - k_{t+1}\right).
\end{aligned}$$

Define, similarly to the Ramsey problem, the saving rate of the social planner as

$$s_t = \frac{k_{t+1}}{(1-\kappa)(1-\alpha)k_t^\alpha}.$$

Then from the first order conditions we obtain

$$
\begin{aligned}
\frac{1}{c_t^y} &= \frac{\beta}{c_{t+1}^o}\alpha k_{t+1}^{\alpha-1} \\
\frac{k_{t+1}}{(k_t^\alpha - k_{t+1})} &= \frac{\alpha\theta k_{t+1}^\alpha}{\left(k_{t+1}^\alpha - k_{t+2}\right)} \\
(1-(1-\kappa)(1-\alpha)s_{t+1}) &= \alpha\theta\left(\frac{1}{(1-\kappa)(1-\alpha)s_t} - 1\right).
\end{aligned}
$$

As in the neoclassical growth model we can show that the only solution to this first order difference equation that does not eventually violate the non-negativity constraint of consumption and does not violate the transversality condition of the social planner is a constant saving rate s solving

$$
(1-(1-\kappa)(1-\alpha)s) = \alpha\theta\left(\frac{1}{(1-\kappa)(1-\alpha)s} - 1\right).
$$

Define $\tilde{s} = (1-\kappa)(1-\alpha)s$ then we have

$$
1-\tilde{s} = \alpha\theta\left(\frac{1}{\tilde{s}} - 1\right)
$$

with solutions $\tilde{s} = 1$ (and thus $s > 1$) and $\tilde{s} = \alpha\theta$. Therefore the constant saving rate that solves the social planner problem from period $t = 1$ onward is given by:

$$
s^{SP} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}.
$$

The optimal sequence of capital stocks, starting from k_0, is therefore given by

$$
\begin{aligned}
k_{t+1} &= (1-\kappa)(1-\alpha)s_t k_t^\alpha \\
&= \alpha\theta k_t^\alpha.
\end{aligned}
$$

Period 0. Let us next characterize the allocation in period $t = 0$. We get

$$
\frac{c_0^o}{c_0^y} = \frac{\beta\omega_{-1}}{\omega_0}
$$

and thus only the ratio of the first two welfare weights matters. Therefore we can, without loss of generality, normalize $\omega_0 = 1$ so that

$$\frac{c_0^o}{c_0^y} = \beta\omega_{-1}.$$

Using this in the resource constraint one obtains

$$c_0^y = \frac{1}{1+\beta\omega_{-1}}\left(k_0^\alpha - k_1\right)$$
$$c_0^o = \frac{\beta\omega_{-1}}{1+\beta\omega_{-1}}\left(k_0^\alpha - k_1\right)$$

$$k_1 = s_0(1-\kappa)(1-\alpha)k_0^\alpha.$$

Then from the first order conditions we get

$$\begin{aligned}
\frac{1}{c_0^y} &= \frac{\beta}{c_1^o}\alpha k_1^{\alpha-1} \\
\frac{k_1\left(1+\beta\omega_{-1}\right)}{\left(k_0^\alpha - k_1\right)} &= \frac{\alpha\left(\theta+\beta\right)k_1^\alpha}{\left(k_1^\alpha - k_2\right)} \\
\frac{s_0(1-\kappa)(1-\alpha)\left(1+\beta\omega_{-1}\right)}{\left(1-s_0(1-\kappa)(1-\alpha)\right)} &= \frac{\alpha\left(\theta+\beta\right)}{\left(1-\alpha\theta\right)}
\end{aligned}$$

and thus

$$\begin{aligned}
s_0^{SP} &= \frac{\alpha\left(\theta+\beta\right)}{(1-\kappa)(1-\alpha)\left[\left(1+\beta\omega_{-1}\right)\left(1-\alpha\theta\right)+\alpha\left(\theta+\beta\right)\right]} \\
&= \frac{\alpha\theta}{(1-\kappa)(1-\alpha)\left[\frac{\left(\theta+\beta\theta\omega_{-1}\right)}{\theta+\beta}\left(1-\alpha\theta\right)+\alpha\theta\right]}.
\end{aligned} \tag{7}$$

Now, suppose that $\omega_{-1} = 1/\theta$. Then (7) simplifies to

$$s_0^{SP} = s^{SP} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}. \tag{8}$$

We summarize these results in the following

Proposition 2. *The solution to the social planner problem, for any $k_0 > 0$, is given by*

$$s_0^{SP} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)\left[\frac{(\theta+\beta\theta\omega_{-1})}{\theta+\beta}(1-\alpha\theta)+\alpha\theta\right]}$$

and associated capital stock in period 1

$$k_1 = s_0^{SP}(1-\kappa)(1-\alpha)k_0^{\alpha}.$$

and consumption allocations in period 0

$$c_0^y = \frac{1}{1+\beta\omega_{-1}}\left(1 - s_0^{SP}(1-\kappa)(1-\alpha)\right)k_0^{\alpha}$$
$$c_0^o = \frac{\beta\omega_{-1}}{1+\beta\omega_{-1}}\left(1 - s_0^{SP}(1-\kappa)(1-\alpha)\right)k_0^{\alpha}$$

and in all periods $t > 0$ by a constant saving rate

$$s^{SP} = \frac{k_{t+1}}{(1-\kappa)(1-\alpha)k_t^{\alpha}} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}$$

and associated sequence of capital stocks

$$k_{t+1} = \alpha\theta k_t^{\alpha} \tag{9}$$

and consumption levels

$$c_t^y = \frac{\theta(1-\alpha\theta)k_t^{\alpha}}{\theta+\beta} \tag{10a}$$
$$c_t^o = \frac{\beta(1-\alpha\theta)k_t^{\alpha}}{\theta+\beta}. \tag{10b}$$

If, in addition $\omega_{-1} = \frac{1}{\theta}$ then $s_0^{SP} = s^{SP}$ and equations (9) and (10) apply for all periods $t \geq 0$.

Also notice that for $\theta = \omega_{-1} = 1$, i.e., for a planner that maximizes steady state utility and also weighs the initial generation equally, then the optimal saving rates in all $t \geq 0$ are

$$s_0^{SP} = s^{SP} = \frac{\alpha}{(1-\kappa)(1-\alpha)} = s^{GR}.$$

We summarize these insights in the next

Corollary 1. *If $\theta = 1$ (associated with maximizing steady state utility), then the social planner chooses the golden rule saving rate*

$$s^{SP} = s^{GR} = \frac{\alpha}{(1-\kappa)(1-\alpha)}$$

in all $t > 0$ and the capital stock converges, in the long run, to its golden rule level

$$k^{GR} = \alpha^{\frac{1}{1-\alpha}}$$

which satisfies

$$\alpha \left[k^{GR}\right]^{\alpha-1} = 1$$

and associated consumption levels

$$\begin{aligned} c^y &= \frac{(1-\alpha)}{1+\beta}\alpha^{\frac{\alpha}{1-\alpha}} \\ c_t^o &= \frac{\beta(1-\alpha)}{1+\beta}\alpha^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Therefore, the social planner chooses the golden rule capital stock k^{GR} maximizing net output $y^{GR} = \left(k^{GR}\right)^{\alpha} - k^{GR}$ and splits it efficiently between c^y and c^o according to the rule $c^o = \beta c^y$. If, in addition, $\omega_{-1} = 1$ then also

$$s_0^{SP} = s^{SP} = \frac{\alpha}{(1-\kappa)(1-\alpha)}.$$

Obviously, the Ramsey equilibrium is not Pareto efficient because it does not provide full consumption insurance against idiosyncratic income risk. What is more remarkable is that even though the optimal Ramsey saving rate is independent of income risk (and the same as in a model where income risk is absent), it is in general different from the saving rate optimally chosen by the social planner (who fully insures the idiosyncratic income risk). This result is summarized in the next

Corollary 2. *For a fixed social discount factor $\theta \in [0, 1]$, the optimal Ramsey saving rate equals the saving rate chosen by the social planner if and only if the following knife edge condition is satisfied:*

$$(1-\kappa) = \frac{\theta(1+\alpha\beta)}{(1-\alpha)(\beta+\theta)}$$

Note that the Ramsey government *can* implement the saving rate desired by the social planner through an appropriate choice of taxes, but unless the condition above is satisfied, it is suboptimal to do so. The reason is that the Ramsey government has no instruments to transfer resources across generations and thus forcing the planner saving rate onto households (by appropriate choice of the capital tax rate) results in an equilibrium allocation of consumption across the young and the old that is typically suboptimal.[3]

E.2 Proof of Constrained Efficiency of Ramsey Allocation

Proof. Define the saving rate of the constrained planner as

$$s_t = \frac{k_{t+1}}{(1-\kappa)MPL(k_t)} = \frac{k_{t+1}}{(1-\alpha)(1-\kappa)k_t^{\alpha}}.$$

Thus, the law of motion for the effective capital stock for the constrained planner is

$$k_{t+1} = s_t(1-\alpha)(1-\kappa)k_t^{\alpha}$$

as in the Ramsey problem. Furthermore, from the constraints on the constrained planner

$$\begin{aligned}
c_t^y &= (1-\kappa)MPL(k_t) - k_{t+1} = (1-s_t)(1-\kappa)(1-\alpha)k_t^{\alpha} \\
c_{t+1}^o(\eta_{t+1}) &= k_{t+1}MPK(k_{t+1}) + \kappa\eta_{t+1}MPL(k_{t+1}) \\
&= \alpha k_{t+1}^{\alpha} + \kappa\eta_{t+1}(1-\alpha)k_{t+1}^{\alpha} \\
&= \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right]k_{t+1}^{\alpha}.
\end{aligned}$$

Thus the consumption allocation is the same as in the Ramsey equilibrium and the solution, in terms of saving rates, of the constrained planner problem is the same as the Ramsey equilibrium. □

[3] Finally note that if one were to treat the social discount factor θ as a *free parameter*, then one concludes that the Ramsey optimal saving rate is efficient, in that it is identical to the choice of the social planner with a *different social discount rate* $\theta^{SP} = \frac{(\beta+\theta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}$.

E.3 Proof of Pareto-Improving Tax-Induced Transition

E.3.1 Log Utility

Proof of Proposition 6. The capital stock evolves according to

$$k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^\alpha.$$

Therefore if the Ramsey government implements s^* through positive capital taxes in the first period of the transition this will lead to a falling capital stock along the transition. Recall from (1) that utility of a generation born in period t is given by

$$V_t = \ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1}))d\Psi.$$

Now, suppose that the policy is implemented (as a surprise) in period 1 where $k_1 = k_0$. The initial old are unaffected by this policy and thus indifferent to the tax reform. Now we need to characterize the utility consequences for all generations born along the transition. Denoting by $s_0 = s^{CE}$ the equilibrium saving rate in the initial steady state, we have

$$\Delta V_t = V_t(s^*) - V_t(s_0) = \ln(c_t^y(s^*)) - \ln(c_t^y(s_0)) + \beta \int \left(\ln(c_{t+1}^o(s^*)) - \ln(c_{t+1}^o(s_0))\right) d\Psi.$$

where the consumption allocations are

$$\begin{aligned} c_t^y(s_t) &= (1-s_t)(1-\kappa)(1-\alpha)k_t^\alpha \\ c_{t+1}^o(\eta_{t+1}; s_t) &= s_t(1-\kappa)(1-\alpha)k_t^\alpha \alpha k_{t+1}^{\alpha-1} + \kappa\eta_{t+1}(1-\alpha)k_{t+1}^\alpha \\ &= \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right] k_{t+1}^\alpha. \end{aligned}$$

Thus

$$\Delta V_t = \underbrace{\ln\left[((1-s^*)k_t^\alpha)\right] - \ln\left[((1-s_0)k_0^\alpha)\right]}_{=\Delta V_t^+} + \alpha\beta\Gamma_2 \left(\underbrace{\ln\left[k_{t+1}\right] - \ln\left[k_0\right]}_{=\Delta V_t^-}\right)$$

Since the capital stock is monotonically decreasing along the transition, $\Delta V_t^- < 0$ for all $t > 0$ and $\Delta V_s^- < \Delta V_t^- < 0$ for all $s > t > 0$, and we call ΔV_t^- the "loss" term. From the monotonically decreasing capital stock it also follows that ΔV_t^+ is monotonically decreasing along the transition. Since in the limit we have $\lim_{t\to\infty} \Delta V_t > 0$ (because s^*

maximizes steady state utility), it follows that $\Delta V_t^+ > 0$ for all $t > 0$ and we therefore refer to ΔV_t^+ as the "gains" term. Finally, since gains are monotonically decreasing and losses—the absolute value $\left|V_t^-\right|$—are monotonically increasing we achieve the smallest gains and largest losses for $t \to \infty$ and since $\lim_{t\to\infty} \Delta V_t > 0$, it follows that $\Delta V_t > 0$ in all $t > 0$. □

E.3.2 Generalization

The previous results generalize to additively separable life-time utility functions of the form

$$V_t = u(c_t^y) + g(c_{t+1}^o, \Psi) \tag{11}$$

with $u' > 0, u'' < 0$ for all $c_t^y > 0$ and $g' > 0, g'' < 0$ for all $c_{t+1}^o > 0$. Aggregating second period consumption with function $g(\cdot)$ nests standard (discounted) expected utility formulations as well as non-expected utility preferences such as Epstein-Zin-Weil preferences, analyzed in Section 6.3.6. As before, write consumption allocations in terms of the saving rate s as $(c_t^y(s), c_{t+1}^o(\eta, s))$. As shorthand, below we denote as $u_s = u'(c_t^y(s)) \times c_t^y(s)'$, with g_s defined correspondingly. Given this notation the first-order condition of the Ramsey problem for $\theta = 1$ is

$$\frac{\partial V_\infty}{\partial s} = u_s + g_s = 0 \qquad \Leftrightarrow \qquad -u_s = g_s. \tag{12}$$

We make the following additional

Assumption 1.

$$\lim_{s\to 1} -u_s > \lim_{s\to 1} g_s \tag{13}$$

and, for all $s \in (\alpha, 1)$,

$$\varepsilon_{u',c} = -\frac{u''(c_t^y(s))}{u'(c_t^y(s))} < \frac{c_t^y(s)''}{c_t^y(s)'} = \varepsilon_{c_s,s}, \tag{14}$$

where $\varepsilon_{u',c}$ is the semi-elasticity of marginal utility[4] with respect to consumption c^y and $\varepsilon_{c_s,s}$ is the semi-elasticity of consumption c^y with respect to the saving rate s.

[4] In in a static stochastic environment this would be equal to the measure of absolute risk aversion. We prefer the term semi-elasticity of marginal utility because first period consumption is not stochastic.

The next proposition generalizes Proposition 6 to additively separable utility functions with the above properties. It also provides conditions for existence and uniqueness of a solution to (12):

Proposition 3. *Let the utility function be given by (11). Under assumption 1 the solution to (12) gives a unique $s^* \in (\alpha, 1)$. Further assume that $s^{CE} > s^*$. Then implementing s^* in period $t = 0$ for all $t \geq 0$ leads to a Pareto improving transition.*

Before proving the above proposition, note that condition (13) is required for existence, and condition (14) for uniqueness of $s^* \in (\alpha, 1)$. We further show that condition (14) implies that $\frac{\partial V_\infty}{\partial s} < 0$ for $s^{CE} > s^*$ so that the generation born in the limit of the transition when the economy approaches the new steady state benefits from implementing $s^* < s^{CE}$. We later establish for Epstein-Zin-Weil preferences, which nest CRRA preferences as a special case, that all these conditions are satisfied. Thus, we show analytically that the conditions apply quite generally. For the general class of HARA utility functions

$$u(c) = \frac{1-\gamma}{\gamma}\left(\frac{\iota \cdot c}{1-\gamma} + \xi\right)^\gamma$$

with parameters $\iota > 0, \xi, \gamma$, and the restriction $\frac{\iota \cdot c}{1-\gamma} + \xi > 0$ and $\gamma \neq 1$ (ruling out linear utility) condition (13) may fail to hold so that there is no solution to the Ramsey problem. For instance, with exponential utility condition (13) may fail to hold since there is no Inada condition as consumption approaches zero, so that $\lim_{s\to 1} -u_s < \infty$.[5]

As for the assumption that $s^{CE} > s^*$ notice that we earlier established that s^{CE} is increasing with risk if there is precautionary savings. Thus, with sufficient risk we have $s^{CE} > s^*$. Also, as for the second part of the proposition on the Pareto improving transition, the proof follows exactly the same logic as the proof of Proposition 6.

This proposition does not address whether the equilibrium has overaccumulated capital. As before, the interesting case is where $s^* < s^{CE} < s^{GR}$, where $s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)}$ is the golden rule saving rate. Finally, notice that the proposition is silent about implementation. We address implementation under the assumption of existence of a unique s^* in the subsequent Proposition 4 for expected utility and later in Proposition 23 for EZW utility.

[5]Consider nested exponential utility, i.e., $\gamma = -\infty$, and $\xi = 1$. Further parameterize $\iota = 1, \alpha = 0.33, \kappa = 0.7$ and $\eta = 1$, i.e., a degenerate deterministic case. Also assume an expected utility formulation with $\beta = 1$

$$g(c^o; \Psi) = \beta \int u(c^o(\eta)) d\Psi(\eta).$$

Then condition (13) fails to hold, an interior s^* does not exist and the optimal saving rate is $s^* = 1$.

Proof of Proposition 3. First, we establish that s^* is unique and that with uniqueness we get for $s^{CE} > s^*$ that $\frac{\partial V_\infty}{\partial s} < 0$. To show this, we analyze the first-order condition of the Ramsey government (12). The next steps will establish that (i) $g_s > 0$ is a continuous and downward sloping function in s, (ii) $-u_s > 0$ for $s > \alpha$, and (iii) that condition (14) is required for a single crossing of g_s and $-u_s$. Findings (i)-(ii) together with (13) establish existence, the additional item (iii) then insures uniqueness of s^*.

Now start from the allocation in the long-run steady state. Recall from Section E.3 above that consumption when young and old is

$$
\begin{aligned}
c_t^y &= (1-s_t)(1-\kappa)(1-\alpha)k_t^\alpha, \\
c_{t+1}^o &= (\alpha+\kappa(1-\alpha)\eta)\, k_{t+1}^\alpha,
\end{aligned}
$$

where

$$
k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^\alpha. \tag{15}
$$

In steady state we thus have

$$
k = (s(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}
$$

and therefore steady state consumption allocations are

$$
c^y = (1-s)s^{\frac{\alpha}{1-\alpha}}\left((1-\alpha)(1-\kappa)\right)^{\frac{1}{1-\alpha}} \tag{16a}
$$

$$
c^o = (\alpha+\kappa(1-\alpha)\eta)\left((1-\alpha)(1-\kappa)\right)^{\frac{\alpha}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}}. \tag{16b}
$$

Use this in the social welfare function with $\theta = 1$ to obtain

$$
\begin{aligned}
V_\infty &= u(c^y) + g(c^o;\Psi) \\
&= u\left((1-s)s^{\frac{\alpha}{1-\alpha}}\left((1-\alpha)(1-\kappa)\right)^{\frac{1}{1-\alpha}}\right) + g((\alpha+\kappa(1-\alpha)\eta)\left((1-\alpha)(1-\kappa)\right)^{\frac{\alpha}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}};\Psi).
\end{aligned}
$$

From the above we readily observe that $g_s > 0$ as well as $g_{ss} < 0$ because of decreasing marginal utility.[6] To establish existence of s^* observe that

$$u_s = u'(c^y(s)) \times c^y(s)' = u'\left((1-\alpha)(1-\kappa)\right)^{\frac{1}{1-\alpha}} \left(-1 + \frac{\alpha}{1-\alpha}(1-s)s^{-1}\right) s^{\frac{\alpha}{1-\alpha}}$$
$$< 0 \qquad \Leftrightarrow \qquad c^y(s)' < 0 \qquad \Leftrightarrow \qquad s > \alpha.$$

because $u'(c^y(s)) > 0$ and thus $u_s < 0$ for $s > \alpha$. If, in addition, condition (13) holds, then there exists at least one solution $s^* \in (\alpha, 1)$. Also notice that condition (13) holds if u satisfies the Inada condition, because then $\lim_{s\to 1} -u_s = \infty$ and $\lim_{s\to 1} g_s < \infty$.

To establish uniqueness we further require that $u'' < 0$ for all $s \in (\alpha, 1)$ so that $-u_s$ is upward sloping and continuous. Observe that

$$u_{ss} = u''(c^y)c^y(s)' + u'(c^y)c^y(s)'' < 0 \qquad \Leftrightarrow \qquad \varepsilon_{u',c} = -\frac{u''(c^y)}{u'(c^y)} < \frac{c^y(s)''}{c^y(s)'} = \varepsilon_{c_s,s}$$

which limits the (positive) semi-elasticity of marginal utility $\varepsilon_{u',c}$ from above. For the semi-elasticity of consumption $\varepsilon_{c_s,s}$ notice that we have already established that $c^y(s)' < 0$ for $s \in (\alpha, 1)$. We next show that for $s \in (\alpha, 1)$ also $c^y(s)'' < 0$ so that $\varepsilon_{c_s,s} > 0$. To see this, write

$$c^y(s)'' = \left((1-\alpha)(1-\kappa)\right)^{\frac{1}{1-\alpha}} \frac{\alpha}{1-\alpha} s^{\frac{\alpha}{1-\alpha}-1} \left[-2 + (1-s)\frac{2\alpha - 1}{1-\alpha}s^{-1}\right]$$

and thus $c^y(s)'' < 0$ if

$$-2 + (1-s)\frac{2\alpha-1}{1-\alpha}s^{-1} < 0 \qquad \Leftrightarrow \qquad s > 2\alpha - 1$$

Before, we have shown that for $s > \alpha$ we have $c^y(s)' < 0$ and since $\alpha > 2\alpha - 1 \Leftrightarrow \alpha < 1$ we know that $s > \alpha$ implies that $c^y(s)'' < 0$ and thus for $s \in (\alpha, 1)$ we get $\frac{c^y(s)''}{c^y(s)'} > 0$. Also, since by property (14) the function $-u_s$ is continuous and upward sloping and since g_s is downward sloping we have that if $s^* \in (\alpha, 1)$ exists, then $s^{CE} > s^*$ implies that $V'_\infty(s) < 0$.

Along the transition, recall that the consumption allocations for generation t is

$$c^y_t(s_t) = (1 - s_t)(1-\kappa)(1-\alpha)k_t^\alpha$$
$$c^o_{t+1}(\eta_{t+1}; s_t) = \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right] k_{t+1}^\alpha.$$

[6] Specifically, we have assumed that $g_c > 0$, $g_{cc} < 0$. Observe from (16b) that $c^o(s)' > 0$ so that $g_s > 0$ and $g_{ss} < 0$.

Thus, assuming a unique $s^* < s^{CE}$ we obtain

$$\Delta V_t = \underbrace{u\left((1-s^*)(1-\kappa)(1-\alpha)k_t^\alpha\right) - u\left((1-s^{CE})(1-\kappa)(1-\alpha)k_0^\alpha\right)}_{=\Delta V_t^+} +$$
$$\underbrace{g\left(\left[\alpha+\kappa\eta_{t+1}(1-\alpha)\right]k_{t+1}^\alpha;\Psi\right) - g\left(\left[\alpha+\kappa\eta_{t+1}(1-\alpha)\right]k_0^\alpha;\Psi\right)}_{=\Delta V_t^-}$$

and since $\frac{\partial c_t^y}{\partial k_t} > 0$ as well as $\frac{\partial c_{t+1}^o}{\partial k_{t+1}} > 0$ the same arguments on the behavior of V_t^+ and V_t^- along the transition as in the proof of Proposition 6 apply. □

E.3.3 Implementation

Observe that the proof above does not say anything about implementation of the saving rates though taxation of capital. The next proposition contains a fairly general implementation result for expected utility. Proposition 23 extends this result to EZW utility.

Proposition 4. *If the utility function in both periods is of the HARA form,*

$$u(c) = \frac{1-\gamma}{\gamma}\left(\frac{\iota c}{1-\gamma}+\xi\right)^\gamma, \tag{17}$$

with parameters $\iota > 0, \xi, \gamma$, $\gamma \neq 1$ such that $\frac{\iota c}{1-\gamma} + \xi > 0$, then in general equilibrium the saving rate s is strictly decreasing in the tax rate τ and any $s^* \in (\alpha, 1]$ can be implemented by a unique (but typically time-dependent) tax rate τ_{t+1}^*.

Proof. Start from the Euler equation for a given period t aggregate wage $w_t = (1-\alpha)k_t^\alpha$

$$u'\left[(1-\kappa)w_t(1-s(\tau_{t+1}))\right] =$$
$$\alpha\beta(1-\tau_{t+1})\left((1-\kappa)w_t\right)^{\alpha-1}s(\tau_{t+1})^{\alpha-1}\int u'\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^\alpha s(\tau_{t+1})^\alpha\right]d\Psi(\eta). \tag{18}$$

Totally differentiate (18) to get

$$
\begin{aligned}
&-(1-\kappa)w_t u''\left[(1-\kappa)w_t(1-s(\tau_{t+1}))\right]\frac{ds(\tau_{t+1})}{d\tau_{t+1}} = \alpha\beta\left((1-\kappa)w_t\right)^{\alpha-1} \\
&\left[-s(\tau_{t+1})^{\alpha-1} + (1-\tau_{t+1})(\alpha-1)s(\tau_{t+1})^{\alpha-2}\frac{ds(\tau_{t+1})}{d\tau_{t+1}}\right]\int u'\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}\right]d\Psi(\eta) \\
&\qquad + \alpha^2\beta(1-\tau_{t+1})\left((1-\kappa)w_t\right)^{2\alpha-1}s(\tau_{t+1})^{2(\alpha-1)}\frac{ds(\tau_{t+1})}{d\tau_{t+1}}\cdot \\
&\qquad \cdot\int u''\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}\right](\alpha+(1-\alpha)\kappa\eta)d\Psi(\eta).
\end{aligned}
$$

Now use the notation

$$
\begin{aligned}
c^y(s(\tau_{t+1})) &= (1-\kappa)w_t(1-s(\tau_{t+1})) \\
c^o(s(\tau_{t+1}),\eta) &= (\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}
\end{aligned}
$$

and divide by $(1-\kappa)w_t$ to rewrite this further as

$$
\begin{aligned}
&-u''\left[c^y(s(\tau_{t+1}))\right]\frac{ds(\tau_{t+1})}{d\tau_{t+1}} = -\alpha\beta\left((1-\kappa)w_t\right)^{\alpha-2} \\
&\quad\left[s(\tau_{t+1})^{\alpha-1} + (1-\tau_{t+1})(1-\alpha)s(\tau_{t+1})^{\alpha-2}\frac{ds(\tau_{t+1})}{d\tau_{t+1}}\right]E\left[u'(c^o(s(\tau_{t+1}),\eta))\right] \\
&\qquad + \alpha^2\beta(1-\tau_{t+1})\left((1-\kappa)w_t\right)^{2(\alpha-1)}s(\tau_{t+1})^{2(\alpha-1)}\frac{ds(\tau_{t+1})}{d\tau_{t+1}}\cdot \\
&\qquad \cdot\int u''\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}\right](\alpha+(1-\alpha)\kappa\eta)d\Psi(\eta).
\end{aligned}
$$

Since $u' > 0$ and $u'' < 0$ ambiguity of implementation may come from the expression

$$
\int u''\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}\right](\alpha+(1-\alpha)\kappa\eta)d\Psi(\eta). \tag{19}
$$

Before proceeding observe that without risk implementation is unambiguous since then

$$
u''\left[(\alpha+(1-\alpha)\kappa)\left[(1-\kappa)w_0\right]^{\alpha}s(\tau)^{\alpha}\right](\alpha+(1-\alpha)\kappa) < 0.
$$

With income risk, observe that with HARA utility (17) we have

$$u' = \iota\left(\frac{\iota c}{1-\gamma} + \xi\right)^{\gamma-1}, \qquad u'' = -\iota^2\left(\frac{\iota c}{1-\gamma} + \xi\right)^{\gamma-2}$$

and thus (19) becomes

$$\begin{aligned}
&-\int \iota^2\left[\frac{\iota}{1-\gamma}(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^\alpha s(\tau_{t+1})^\alpha + \xi\right]^{\gamma-2}(\alpha+(1-\alpha)\kappa\eta)d\Psi(\eta)\\
&= -\iota^2\int\left[\left(\frac{\iota}{1-\gamma}(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^\alpha s(\tau_{t+1})^\alpha + \xi\right)(\alpha+(1-\alpha)\kappa\eta)^{\frac{1}{\gamma-2}}\right]^{\gamma-2} d\Psi(\eta)\\
&= -\iota^2\int\left[\left(\frac{\iota}{1-\gamma}(\alpha+(1-\alpha)\kappa\eta)^{\frac{\gamma-1}{\gamma-2}}\left[(1-\kappa)w_t\right]^\alpha s(\tau_{t+1})^\alpha + \xi(\alpha+(1-\alpha)\kappa\eta)^{\frac{1}{\gamma-2}}\right)\right]^{\gamma-2} d\Psi(\eta)\\
&= \Lambda(s(\tau_{t+1}); \iota, \xi, \alpha, \kappa, \gamma, \eta) < 0
\end{aligned}$$

and thus for HARA preferences defined in the proposition s_t and τ_{t+1} are strictly negatively related, implying that for any saving rate there exists a unique tax rate implementing this saving rate as a competitive equilibrium. □

E.3.4 Marginal Reforms

The next corollary studies marginal tax reforms rather than implementing the full Ramsey equilibrium.

Corollary 3. *Let Assumption 1 hold and assume that $s^{CE} > s^*$. Implementing a saving rate $s^{CE} - \epsilon \geq s^*$ for small $\epsilon > 0$ in all periods $t \geq 0$ through a time-varying tax rate τ_{t+1} yields a Pareto improvement.*

Proof. Replace in the proof of Proposition 3 s^* by $s^{CE} - \epsilon \geq s^*$ to note that the same arguments on monotone transitions of the gains and loss terms apply. □

E.4 Savings Subsidy Does Not Induce Pareto Improvement

In this section we show, in contrast to the previous section, that even if $s^{CE} < s^*$, implementing the Ramsey (for $\theta = 1$) saving rate s^* through a savings subsidy $\tau^* < 0$ does *not* lead to a Pareto improving transition. We exploit the fact that in the first period of the transition the capital stock $k_1 = k_0$ is predetermined, and the capital stock in $t = 2$ satisfies

$$k_2 = s(1-\alpha)(1-\kappa)k_0^\alpha$$

for any saving rate implemented by a given tax policy. Then we can calculate lifetime utility of the first transition generation, as a function of an implemented saving rate s, as

$$\begin{aligned} V_1(s) &= \ln\left((1-s)(1-\kappa)(1-\alpha)k_0^\alpha\right) + \beta \int \ln\left(\alpha + \kappa\eta_2(1-\alpha)\right)\left(s(1-\alpha)(1-\kappa)k_0^\alpha\right)^\alpha d\Psi(\eta) \\ &= \ln(1-s) + \beta\alpha\ln(s) + \ln\left((1-\kappa)(1-\alpha)k_0^\alpha\right) \\ &+ \beta \int \ln\left(\alpha + \kappa\eta_2(1-\alpha)\right)\left((1-\alpha)(1-\kappa)k_0^\alpha\right)^\alpha d\Psi(\eta) \end{aligned}$$

and thus

$$\begin{aligned} V_1'(s) &= -\frac{1}{1-s} + \frac{\alpha\beta}{s} \\ V_1''(s) &= -\frac{1}{(1-s)^2} - \frac{\alpha\beta}{s^2} < 0. \end{aligned}$$

Therefore $V_1(s)$ is strictly concave in s. Therefore, if $V_1'(s = s^{CE}) \leq 0$, then $V(s = s^{CE}) > V(s)$ for all $s > s^{CE}$. We have

$$\begin{aligned} V_1'(s = s^{CE}) = -\frac{1}{1-s^{CE}} + \alpha\beta\frac{1}{s^{CE}} \leq 0 \\ \Leftrightarrow \qquad s^{CE} \geq \frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

which is satisfied, exploiting expression (11) for the optimal competitive equilibrium saving rate (with zero taxes). Thus not only is implementing $\tau^* < 0$ not Pareto improving if $s^{CE} < s^*$, but in fact any policy reform that induces a saving rate in period 1 above the competitive saving rate with zero taxes, s^{CE}, will not result in a Pareto improvement, since it makes the first generation strictly worse off.

F Endogenous Labor Supply and Labor Income Taxation

In this section we provide the details of the analysis of the model with endogenous labor supply.

F.1 Model Setup

As discussed in the main text, we assume households have lifetime utility defined over consumption when young, and stochastic consumption and labor allocations when old $c_t^y, c_{t+1}^o(\eta), l_{t+1}^o(\eta)$ given by

$$V_t = \ln(c_t^y) + \beta \int \left[\ln(c_{t+1}^o(\eta)) + \gamma \ln(1 - l_{t+1}^o(\eta))\right] d\Psi(\eta), \tag{20}$$

where $\gamma > 0$ is a parameter.[7] The budget constraints of a typical generation now read as

$$\begin{aligned} c_t^y + a_{t+1} &= (1-\kappa)w_t && \text{(21a)} \\ c_{t+1}^o(\eta) &= (1-\tau_{t+1})R_{t+1}a_{t+1} + \kappa\eta(1-\tau_{t+1}^l)w_{t+1}l_{t+1}^o(\eta) + T_{t+1}, && \text{(21b)} \end{aligned}$$

where τ_{t+1}^l is the proportional labor income tax rate. As before, tax revenues from capital and labor income taxes on the old are rebated to them in a lump-sum fashion:

$$T_{t+1} = \tau_{t+1}R_{t+1}a_{t+1} + \tau_{t+1}^l \kappa w_{t+1} \int \eta l_{t+1}^o(\eta) d\Psi(\eta). \tag{22}$$

The production technology transforming capital and labor (K_t, L_t) into output and the associated firm problem remains unchanged, and equilibrium in the labor market obtains if

$$L_t = 1 - \kappa + \kappa \int \eta l_t^o(\eta) d\Psi(\eta). \tag{23}$$

F.2 Definition of Competitive Equilibrium

Definition 1. *For a given initial capital stock K_0 and a given sequence of tax rates $\{(\tau_t^l, \tau_t)\}$ a competitive equilibrium is a sequence of allocations for households, $\{(c_t^y, a_{t+1}, c_{t+1}^o(\eta), l_{t+1}^o(\eta))\}$, a sequence of allocations for firms, $\{(K_t, L_t)\}$, a sequence of factor prices $\{(R_t, w_t)\}$ and a sequence of transfers $\{T_t\}$ such that*

1. *for all t ≥ 0, given $(\tau_{t+1}^l, \tau_t, T_t)$ and (w_t, w_{t+1}, R_{t+1}) the allocation $(c_t^y, a_{t+1}, c_{t+1}^o(\eta), l_{t+1}^o(\eta))$ maximizes (1) subject to $(21a)$ and $(21b)$;*

[7] The utility function of the initial old is similarly defined.

2. *given* (τ_0^l, τ_0, T_0) *and* (w_0, R_0) *the allocation* $(c_0^o(\eta), l_0^o(\eta))$ *maximizes*

$$V_{-1} = \int [\ln(c_0^o(\eta)) + \gamma(1 - l_0^o(\eta))] d\Psi(\eta) \tag{24}$$

subject to

$$c_0^o(\eta) = (1 - \tau_0) R_0 a_0 + \kappa\eta(1 - \tau_0^l) w_0 l_0^o(\eta) + T_0; \tag{25}$$

3. *For all* $t \geq 0$, *factor prices satisfy*

$$R_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha} \tag{26a}$$

$$w_t = (1 - \alpha) K_t^{\alpha} L_t^{-\alpha}; \tag{26b}$$

4. *for all* $t \geq 0$, *the government budget constraint is satisfied:*

$$T_t = \tau_t R_t K_t + \tau_t^l \kappa w_t \int l_t^o(\eta) \eta d\Psi(\eta); \tag{27}$$

5. *the markets for labor, capital and final goods clear in every period: for all* $t \geq 0$*:*

$$L_t = 1 - \kappa + \kappa \int l_t^o(\eta) \eta d\Psi(\eta) \tag{28}$$

$$K_{t+1} = a_{t+1} \tag{29}$$

$$F(K_t, L_t) = c_t^y + \int c_t^o(\eta) d\Psi(\eta) + K_{t+1}. \tag{30}$$

F.3 Analysis

We now show that the optimal saving rate the Ramsey government chooses is identical to the one in Proposition 2, and is implemented with a tax on capital that is strictly increasing in idiosyncratic labor productivity risk η and strictly decreasing in the social welfare weights on future generations. The analysis proceeds in two steps, we first show that a given sequence of private aggregate saving rates $s_t = \frac{K_{t+1}}{(1-\kappa)w_t} \in (0, 1)$ and aggregate labor $L_t \geq 0$ can be implemented by choice of a sequence of capital and labor income tax rates (τ_{t+1}, τ_t^l). This implies that the Ramsey government can directly choose sequences $\{s_t, L_t\}$ and then implement them as competitive equilibrium with proportional capital and labor income taxes. Second, we show that the optimal Ramsey saving rate is identical to the one with exogenous labor supply, and the capital tax rate implementing it has the same form as before.

F.3.1 Competitive Equilibrium for Given Tax Policy

First, we characterize the competitive equilibrium for a given sequence of capital and labor income tax rates. We state the household optimality conditions, and then show how to aggregate them, exploiting the market clearing conditions and the government budget constraint in general equilibrium.

Optimal Household Decisions For given factor prices and tax policies, the household makes a labor-leisure choice and a consumption savings choice. The next lemma characterizes this choice.

Lemma 1. *Let assumption 1 hold, and assume that the allocations are interior.*[8] *Then the optimal choice of the saving rate* $s_t = \frac{a_{t+1}}{(1-\kappa)w_t}$ *and stochastic old-age labor supply and consumption* $(l^o_{t+1}(\eta), c^o_{t+1}(\eta))$ *are given by*

$$l^o_{t+1}(\eta) = 1 - \frac{\gamma c^o_{t+1}(\eta)}{\kappa\eta w_{t+1}(1-\tau^l_{t+1})} \tag{31}$$

$$1 = \beta(1-\tau_{t+1}) \int \left(\frac{1-s_t}{s_t(1-\tau_{t+1}) + \frac{\kappa w_{t+1}\eta l^o_{t+1}(\eta)(1-\tau^l_{t+1})}{(1-\kappa)w_t R_{t+1}} + \frac{T_{t+1}}{(1-\kappa)w_t R_{t+1}}} \right) d\Psi(\eta) \tag{32}$$

$$c^o_{t+1}(\eta) = (1-\tau_{t+1})R_{t+1}s_t(1-\kappa)w_t + \kappa\eta(1-\tau^l_{t+1})w_{t+1}l^o_{t+1}(\eta) + T_{t+1}. \tag{33}$$

General Equilibrium Now we aggregate the individual decisions and express aggregate labor supply and the aggregate private saving rate as a function of the policy instruments. Aggregate labor supply of the old, L^o_t, and thus total aggregate labor supply L_t are given by

$$L^o_t = \int l^o_t(\eta)\eta d\Psi(\eta)$$
$$L_t = 1 - \kappa + \kappa L^o_t$$

[8] Consumption and leisure are strictly positive almost surely by the Inada conditions implied by log-utility. However, labor supply might optimally be equal to zero for sufficiently low η. We will below state a sufficient condition on the support of η such that labor supply is indeed interior η-almost surely.

and factor prices are determined as

$$w_t = (1-\alpha)K_t^{\alpha}L_t^{-\alpha}$$
$$R_t = \alpha K_t^{\alpha-1}L_t^{1-\alpha}.$$

Aggregate transfers are given by

$$T_{t+1} = \tau_{t+1}R_{t+1}K_{t+1} + \tau_{t+1}^{l}\kappa w_{t+1}L_{t+1}^{o}. \tag{34}$$

Finally, the aggregate capital stock, the endogenous state variable in this model, evolves as a function of the saving rate $s_t = \frac{K_{t+1}}{(1-\kappa)w_t} = \frac{a_{t+1}}{(1-\kappa)w_t}$

$$K_{t+1} = a_{t+1} = s_t(1-\kappa)w_t = s_t(1-\kappa)(1-\alpha)K_t^{\alpha}L_t^{-\alpha}. \tag{35}$$

Individual and Aggregate Labor Supply In order to obtain a tractable expression for aggregate labor supply, first insert the budget constraint when old (33) into the optimality condition for individual labor supply, equation (31). This delivers, after rearranging,

$$(1+\gamma)(1-\tau_{t+1}^{l})\eta l_{t+1}^{o}(\eta) = \eta(1-\tau_{t+1}^{l}) - \frac{\gamma\left[(1-\tau_{t+1})R_{t+1}s_t(1-\kappa)w_t + T_{t+1}\right]}{\kappa w_{t+1}}. \tag{36}$$

Now we can aggregate both sides of this equation by integrating with respect to idiosyncratic productivity shocks, to obtain

$$(1+\gamma)L_{t+1}^{o} = 1 - \frac{\gamma\left[(1-\tau_{t+1})R_{t+1}s_t(1-\kappa)w_t + T_{t+1}\right]}{(1-\tau_{t+1}^{l})\kappa w_{t+1}}$$

and exploiting the expression for aggregate wages, interest rates and transfers we obtain

$$(1+\gamma)L_{t+1}^{o} = 1 - \frac{\gamma\alpha\left(1-\kappa+\kappa L_{t+1}^{o}\right)}{(1-\tau_{t+1}^{l})\kappa(1-\alpha)} - \frac{\gamma\tau_{t+1}^{l}L_{t+1}^{o}}{(1-\tau_{t+1}^{l})}$$

which yields aggregate equilibrium labor L_{t+1}^{o} and thus L_{t+1} solely as a function of the labor income tax rate τ_{t+1}^{l}

$$L^o_{t+1}(\tau^l_{t+1}) \;=\; \int \eta l^o_{t+1}(\eta)d\Psi(\eta) = \frac{1-\tau^l_{t+1} - \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{1-\tau^l_{t+1}+\frac{\gamma}{1-\alpha}} \tag{37a}$$

$$L_{t+1}(\tau^l_{t+1}) \;=\; 1-\kappa+\kappa L^o_{t+1}(\tau^l_{t+1}) = 1-\kappa+\kappa\frac{1-\tau^l_{t+1} - \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{1-\tau^l_{t+1}+\frac{\gamma}{1-\alpha}}. \tag{37b}$$

Crucially, aggregate equilibrium labor supply is independent of the saving rate and the tax rate on capital shaping the dynamics of the economy, and exclusively depends on the labor income tax rate when old.[9] Define $\bar{\tau}^l_{t+1} = \frac{\kappa(1-\alpha)-\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)} < 1$. We have the following

Proposition 5. *For any tax rate* $\tau^l_{t+1} \in \left(-\infty, \bar{\tau}^l_{t+1}\right)$ *aggregate labor supply is given by equation* (37b). *Aggregate labor supply is strictly decreasing in the labor income tax rate* τ^l_{t+1}. *Thus, for any aggregate labor supply* $L^o_{t+1} \in (0,1)$ *there exists a unique labor income tax rate* τ^l_{t+1} *that implements this* L^o_{t+1} *as part of a competitive equilibrium.*

Note that the labor income tax rate implementing any positive labor supply of the old is strictly less than 1. Optimal individual labor supply $l^o_{t+1}(\eta;\tau^l_{t+1})$ (see equation (38)) is only a function of the labor income tax rate and the idiosyncratic shock. The fact that aggregate labor supply is strictly decreasing in the labor income tax rate follows from the fact that

$$L^o_t = \frac{1-\tau^l_t - \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{1-\tau^l_t+\frac{\gamma}{1-\alpha}}$$

and thus, taking the derivative with respect to the labor income tax rate τ^l_t yields:

$$\begin{aligned}\frac{\partial L^o_t}{\partial \tau^l_t} =& \frac{-(1-\tau^l_t+\frac{\gamma}{1-\alpha})+1-\tau^l_t-\frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{(1-\tau^l_t+\frac{\gamma}{1-\alpha})^2}\\ =& -\frac{\frac{\gamma}{1-\alpha}+\frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{(1-\tau^l_t+\frac{\gamma}{1-\alpha})^2} < 0.\end{aligned}$$

We also note that we can express individual labor supply exclusively in terms of individual labor productivity and labor income taxes, independent of the saving rate and

[9]We assume that the utility weight on leisure γ is sufficiently small and/or the productivity κ of the old sufficiently large that the Ramsey government finds it optimal to implement positive aggregate labor supply of the old. Otherwise old households do not work, and idiosyncratic risk trivially becomes irrelevant. A tax rate $\tau^l_{t+1} < \bar{\tau}^l_{t+1}$ insures this.

independent of the tax rate on capital. Rewriting (36) yields

$$\begin{aligned} l^o_{t+1}(\eta) &= \frac{1}{(1-\tau^l_{t+1})(1+\gamma)}\left(1-\tau^l_{t+1}-\frac{\gamma\alpha L_{t+1}}{\eta\kappa(1-\alpha)}-\frac{\gamma\tau^l_{t+1}L^o_{t+1}}{\eta}\right) \\ &= \frac{1}{(1-\tau^l_{t+1})(1+\gamma)}\left(1-\tau^l_{t+1}-\frac{\gamma\alpha(1-\kappa)+[\alpha+(1-\alpha)\tau^l_{t+1}]\gamma\kappa L^o_{t+1}}{\eta\kappa(1-\alpha)}\right) \\ &= l^o_{t+1}(\eta;\tau^l_{t+1}), \end{aligned} \tag{38}$$

where we note that $L^o_{t+1} = L^o_{t+1}(\tau^l_{t+1})$. We can now also state a condition to insure that individuals find it optimal to supply positive labor even at the lowest productivity level $\underline{\eta}$. For this we need

$$\underline{\eta} > \gamma\frac{\alpha\,(1-\kappa)+\left[\alpha+(1-\alpha)\tau^l_{t+1}\right]\kappa\left(\frac{1-\tau^l_{t+1}-\frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{1-\tau^l_{t+1}+\frac{\gamma}{1-\alpha}}\right)}{(1-\tau^l_{t+1})\kappa(1-\alpha)} = \Xi(\gamma,\tau^l_{t+1}). \tag{39}$$

We note that since $\Xi(\gamma = 0, \tau^l_{t+1}) = 0$, by continuity in γ for every $\tau^l_{t+1} \in (-\infty, 1)$ there exists a small enough γ such that this condition is satisfied and labor supply is positive for every possible productivity level. We therefore make

Assumption 2. *The lower bound of the productivity shock $\underline{\eta}$ satisfies equation (39) for all $\tau^l_{t+1} \leq \bar{\tau}^l_{t+1}$.*

With this result in hand we can now proceed to obtain the competitive equilibrium saving rate as in the benchmark model with exogenous labor supply.

The Aggregate Saving Rate We can now express the saving rate in (32) as a function of the allocation of labor, which we have shown in the previous subsection just to depend on the labor income tax rate τ^l_{t+1}. Using (32) and the expressions for wages, interest rates and transfers in general equilibrium yields

$$1 = \alpha\beta(1-\tau_{t+1})\left(\frac{1-s_t}{s_t}\right)\int\left(\frac{1}{\alpha+\frac{(1-\alpha)\kappa}{L_{t+1}(\tau^l_{t+1})}\left[(1-\tau^l_{t+1})\eta l^o_{t+1}(\eta;\tau^l_{t+1})+\tau^l_{t+1}L^o_{t+1}(\tau^l_{t+1})\right]}\right)d\Psi(\eta)$$

which gives the following equilibrium saving rate

$$s_t = \frac{1}{1+\left[\alpha\beta(1-\tau_{t+1})\Gamma(\alpha,\kappa;\tau^l_{t+1},\Psi)\right]^{-1}} = s(\alpha,\kappa;\tau_{t+1},\tau^l_{t+1},\Psi), \tag{40}$$

where

$$\Gamma(\alpha,\kappa;\tau^l_{t+1},\Psi)=\int\left(\frac{1}{\alpha+\frac{(1-\alpha)\kappa}{L_{t+1}(\tau^l_{t+1})}\left[(1-\tau^l_{t+1})\eta l^o_{t+1}(\eta;\tau^l_{t+1})+\tau^l_{t+1}L^o_{t+1}(\tau^l_{t+1})\right]}\right)d\Psi(\eta) \tag{41}$$

completely summarizes the impact of idiosyncratic productivity and thus income risk on the optimal saving decision. Note that the labor income tax rate affects the risk term Γ through its impact on individual and aggregate labor supply. However, since for every labor income tax rate τ^l_{t+1} satisfying the restriction in proposition 5 the term Γ is a positive constant, we immediately have the following

Proposition 6. *For any labor income tax rate* $\tau^l_{t+1}\in\left(-\infty,\frac{\kappa(1-\alpha)-\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}\right)$ *and any tax rate on capital* $\tau_{t+1}\in(-\infty,1)$ *the aggregate equilibrium saving rate* $s_t\in(0,1)$ *is given in equation* (40). *Consequently, for any saving rate* $s_t\in(0,1)$ *and given a labor income tax rate and associated labor allocation there exists a unique capital tax rate* $\tau_{t+1}\in(-\infty,1)$ *that implements this saving rate as part of a competitive equilibrium.*

The previous two propositions demonstrate the sequential nature of solving for the competitive equilibrium, given tax policy. In each period $t\geq 0$, given a labor income tax rate τ^l_t, we can solve for equilibrium labor supply $(l^o_t(\eta),L^o_t,L_t)$. Then, given this labor allocation, which in turn determines $\Gamma(\alpha,\kappa;\tau^l_{t+1},\Psi)$, and given a tax on capital τ_{t+1}, one solves for the equilibrium saving rate s_t. Finally, the capital stock K_t and the saving rate s_t today determine the aggregate capital stock in period $t+1$. Thus, given an initial condition K_0, any aggregate allocation of labor and savings $\{L_t,s_t\}$ and associated allocation of individual labor $\{l^o_t(\eta)\}$ and capital $\{K_{t+1}\}$ in equation (35) can be implemented as a competitive equilibrium through a suitable choice of labor income and capital tax rates $\{\tau^l_t,\tau_{t+1}\}$.

F.3.2 Optimal Ramsey Allocations and Tax Policy

The objective of the government is to maximize social welfare, as in equation (4), by choice of capital taxes $\{\tau_t\}_{t=0}^{\infty}$ and labor taxes $\{\tau^l_t\}_{t=0}^{\infty}$ and where V_t is lifetime utility of generation t in the competitive equilibrium associated with the sequence $\{\tau_t,\tau^l_t\}_{t=0}^{\infty}$. From the previous implementation result we know that the Ramsey government can, for any $t\geq 0$, implement any desired aggregate labor supply allocation L^o_t,L_t with an appropriate choice of labor income taxes τ^l_t. Given these choices it can then implement any aggregate saving rate s_t with an appropriate choice of τ_{t+1}. Note that since the initial old already made their savings decisions and the revenue from the capital tax is lump-sum distributed to them, the tax rate τ_0 is irrelevant for welfare. We now express expected lifetime utility of a given generation directly in terms of aggregate allocations; the Ramsey government chooses these allocations to maximize social welfare and implements these allocations as a competitive equilibrium with taxes, as discussed above. Lifetime utility of generation t can be expressed purely as a function of the beginning of the period capital stock, and the aggregate saving rate and aggregate labor supply when young and when old:

$$V_t = V(K_t, s_t, L_t^o, L_{t+1}^o) = u((1-s_t)(1-\kappa)(1-\alpha) K_t^\alpha L_t(L_t^o)^{-\alpha}) + \beta \int u\Big(\kappa w(s_t, L_t^o, L_{t+1}^o) \cdot$$
$$\Big[\eta l_{t+1}^o(\eta, L_{t+1}^o)(1-\tau_{t+1}^l(L_{t+1}^o)) + \tau_{t+1}^l(L_{t+1}^o)L_{t+1}^o\Big] + R(s_t, L_t^o, L_{t+1}^o)K_{t+1}(s_t, L_t^o), l_{t+1}^o(\eta, L_{t+1}^o)\Big) d\Psi(\eta)$$

where the aggregate components are themselves given by

$$L_t(L_t^o) = 1-\kappa+\kappa L_t^o \tag{42a}$$
$$L_{t+1}(L_{t+1}^o) = 1-\kappa+\kappa L_{t+1}^o \tag{42b}$$
$$\tau_{t+1}^l(L_{t+1}^o) = \frac{\kappa(1-\alpha-L_{t+1}^o(1+\gamma-\alpha)) - \gamma\alpha(1-\kappa)}{\kappa(1-\alpha)(1-L_{t+1}^o)} \tag{42c}$$
$$l_{t+1}^o(\eta, L_{t+1}^o) = \frac{\left(1-\tau_{t+1}^l(L_{t+1}^o) - \frac{\gamma\alpha L_{t+1}(L_{t+1}^o)}{\eta\kappa(1-\alpha)} - \frac{\gamma\tau_{t+1}^l L_{t+1}^o}{\eta}\right)}{(1-\tau_{t+1}^l(L_{t+1}^o))(1+\gamma)} \tag{42d}$$
$$K_{t+1}(s_t, L_t^o) = s_t(1-\kappa)(1-\alpha)K_t^\alpha L_t(L_t^o)^{-\alpha} \tag{42e}$$
$$w(s_t, L_t^o, L_{t+1}^o) = (1-\alpha)\left[K_{t+1}(s_t, L_t^o)\right]^\alpha L_{t+1}(L_{t+1}^o)^{-\alpha} \tag{42f}$$
$$R(s_t, L_t^o, L_{t+1}^o) = \alpha\left[K_{t+1}(s_t, L_t^o)\right]^{\alpha-1} L_{t+1}(L_{t+1}^o)^{1-\alpha}. \tag{42g}$$

Similarly, remaining lifetime utility of the initial old (and already substituting out factor prices) is given by

$$V_{-1} = V(K_0, L_0^o) =$$
$$\beta \int u\Big(\kappa(1-\alpha)K_0^\alpha L_0(L_0^o)^{-\alpha}\left[\eta l_0^o(\eta, L_0^o)(1-\tau_0^l(L_0^o)) + \tau_0^l(L_0^o)L_0^o\right] + \alpha K_0^\alpha L_0(L_0^o)^{1-\alpha}, l_0^o(\eta, L_0^o)\Big) d\Psi(\eta).$$

Exploiting the assumption of logarithmic utility in consumption and leisure, the objective of the Ramsey government (including the initial generation) can be written as

$$
\begin{aligned}
W(K_0) &= \sum_{t=-1}^{\infty} \omega_t V_t \\
&= \omega_{-1}\beta \int \big[\alpha \log(K_0) - \alpha \log(L_0(L_0^o)) + \log(\kappa(1-\alpha)[\eta l_0^o(\eta, L_0^o)(1-\tau_0^l(L_0^o)) + \tau_0^l L_0^o] + \alpha L_0(L_0^o)) \\
&+ \gamma \log(1 - l_0^o(\eta))\big] d\Psi(\eta) \\
&+ \sum_{t=0}^{\infty} \omega_t \Bigg[\log(1-s_t) + \log(1-\kappa) + \log(1-\alpha) + \alpha \log(K_t) - \alpha \log(L_t(L_t^o)) \\
&+ \beta \int \Big(\alpha \log(s_t) + \alpha \log(1-\kappa) + \alpha \log(1-\alpha) + \alpha^2 \log(K_t) - \alpha^2 \log(L_t(L_t^o)) - \alpha \log(L_{t+1}((L_{t+1}^o))) \\
&+ \log[\kappa(1-\alpha)(\eta l_{t+1}^o(\eta, (L_{t+1}^o))(1 - \tau_{t+1}^l((L_{t+1}^o))) + \tau_{t+1}^l(L_{t+1}^o) L_{t+1}^o) + \alpha L_{t+1}(L_{t+1}^o)] \\
&+ \gamma \log(1 - l_{t+1}^o(\eta), L_{t+1}^o) \Big) d\Psi(\eta) \Bigg],
\end{aligned}
$$

where the log-capital stock $\log(K_t)$ can be expressed as

$$
\begin{aligned}
\log(K_t) &= (\log(1-\alpha) + \log(1-\kappa)) \left(\frac{1-\alpha^t}{1-\alpha} \right) + \alpha^t \log(K_0) \\
&+ \sum_{\tau=1}^{t} \alpha^{\tau-1} \log(s_{t-\tau}) - \sum_{\tau=1}^{t} \alpha^{\tau-1} \log(L_{t-\tau}) \\
&= \log K_t(K_0, \{s_\tau, L_\tau\}_{\tau=0}^{t-1}).
\end{aligned}
$$

Thus we note that the objective function can be written purely in terms of the aggregate allocations $\{s_t, L_t^o\}_{t=0}^{\infty}$ and that it is additively separable in time between the savings rate s_t on one hand and aggregate labor supply L_t^o on the other hand. This in turn will greatly facilitate the characterization of the optimal Ramsey allocations.

Optimal Saving Rate Ignoring constants that are irrelevant for maximization *with respect to the savings rate* s_t, this part $W^s(K_0)$ of the social welfare function can be expressed as:

$$
\begin{aligned}
W^s(K_0) &= \sum_{t=0}^{\infty} \omega_t \left[\log(1-s_t) + \alpha\beta \log(s_t) + \alpha(1+\alpha\beta)\log(K_t)\right] \\
&= \sum_{t=0}^{\infty} \omega_t \left[\log(1-s_t) + \log(s_t) \left(\alpha\beta + \alpha(1+\alpha\beta) \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} \right) \right].
\end{aligned}
$$

Taking first order conditions with respect to s_t and setting it to zero then immediately results in the following Proposition 7, with the implementing capital tax rate directly implied by equation (40).

Proposition 7. *The optimal Ramsey saving rate with endogenous labor supply is given by*

$$s_t = \frac{1}{1 + \left(\alpha\beta + \alpha\left(1 + \alpha\beta\right)\sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}} \in (0,1). \tag{43}$$

Furthermore, if we assume that relative social welfare weights are constant, $\frac{\omega_{t+1}}{\omega_t} = \theta$ *for all* t*, then the optimal Ramsey saving rates in* (43) *are constant over time and given by*

$$s_t = s = \frac{\alpha(\beta + \theta)}{1 + \alpha\beta} \in (0,1).$$

The optimal tax on capital implementing this saving rate as competitive equilibrium is given by equation (40), *or explicitly, as*

$$1 - \tau_{t+1} = \frac{s_t}{\left(1 - s_t\right)\alpha\beta\Gamma(\alpha, \kappa; \tau^l_{t+1}, \Psi)}, \tag{44}$$

where $\Gamma(\alpha, \kappa; \tau^l_{t+1}, \Psi)$ *was defined in equation* (41).

This result demonstrates that the optimal saving rate s_t chosen by the Ramsey planner is independent of the optimal labor allocation and the labor income tax rates that implement them, and identical to the one with exogenous labor supply. It again is independent of the extent of idiosyncratic labor income risk. The optimal tax on capital τ_{t+1} is strictly increasing in income risk Γ, and depends on the optimal labor allocation determining this risk, and thus on the optimal labor income tax rate that governs it.[10] Crucially, since this optimal rate is less than one, idiosyncratic labor income risk continues to be present, and the precautionary savings channel and associated pecuniary externality remains operational.

F.4 Details of the Quantitative Analysis

To get a sense of the extent to which endogenous labor supply and labor income taxation affects the optimal tax on capital quantitatively, we extend the calibration from Section 4.3

[10] The optimal allocation of labor L_{t+1} is determined from a static first order condition of the Ramsey problem which has no closed-form solution but is straightforward to solve numerically. Proposition 5 then gives the associated optimal labor income tax rate implementing this labor allocation.

to endogenous labor supply. Apart from the idiosyncratic productivity process we keep all parameters the same, and we recalibrate the distribution of η, together with the new leisure utility parameter γ in such a way that minimum, mean and log-variance of labor income is the same as in the benchmark economy, and average hours worked are 1/3 of total time. Table 1 contains the resulting parameter values, Figure 2 (in the main text) plots the optimal capital and labor income tax rates against the (annualized) social discount factor and Table 2 summarizes optimal policies for $\theta = 0.995$ and $\theta = 1$.

Table 1: Parameter Values: Endogenous Labor Supply

Parameter		Exo. Labor	Target	End. Labor
γ	Labor utility weight	0	$E[l(\eta)] = 0.3333$	0.73
$\underline{\eta}$	η support lower bound	0.05	$\underline{\eta} l(\underline{\eta})/\eta_{med} l(\eta_{med})$ = 3.35%	1.0
σ_η	η-risk	0.81	$Var[\log(\eta l(\eta))]$ = 0.648	0.45
$E[\eta]$	Mean idiosyn. shock	1	$E[\eta l(\eta)] = 1$	2.68

Notes:. This table summarizes the endogenously calibrated parameters used for the quantitative analysis of the model with endogenous labor supply. Parameters α, κ and the initial tax rate τ^k used in calibration are given in Table 1.

Table 2: Annualized Interest Rate [in %] and Optimal Tax Rates [in %]

Parameter Configuration	Interest Rate (Initial CE)	Optimal Capital Tax Rate τ^*	Opt. Cap. Income Tax Rate τ^{k^*}
	Annualized Discount Factor $\theta = 0.995$		
V[log($\eta l(\eta)$)]=0.6480	0.57%	5.11%	31.73%
V[log($\eta l(\eta)$)]=0	0.80%	-3.06%	-19.00%
	Annualized Discount Factor $\theta = 1.0$		
V[log($\eta l(\eta)$)]=0.6480	0.58%	-6.24%	-63.80%
V[log($\eta l(\eta)$)]=0	0.80%	-14.90%	-152.27%

Notes: This table shows the equilibrium interest rate at the initial competitive equilibrium, the optimal capital tax rate τ^*, and the optimal capital income tax rate τ^{k^*} for the stochastic and the deterministic economy with endogenous labor supply. Corresponding results for the model with exogenous labor supply were shown in Table 2.

G Intergenerational Redistribution

G.1 Pension System

G.1.1 Setup

The budget constraints of households under a time varying capital tax τ_t, a time varying contribution rate to the pension system τ_t^p and a flat pension payment b_{t+1} in the two periods of life are

$$\begin{aligned} c_t^y + a_{t+1} &= w_t(1-\kappa)(1-\tau_t^p) \\ c_{t+1}^o(\eta) &= a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta w_{t+1} + b_{t+1} + T_{t+1}. \end{aligned}$$

We assume a PAYG pension system (balanced budget) so that

$$\tau_t^p(1-\kappa)w_t = b_t.$$

Furthermore, while our formal analysis also encompasses the case of an unrestricted pay-as-you-go pension system, we are mainly interested in a scenario where pension payments are restricted to be positive (i.e., there is no reverse pension system). In this scenario the constraint $\tau_t^p \geq 0$ applies.

Finally, as in the main text, the budget constraint of the capital tax system is

$$\tau_t R_t a_t = T_t.$$

G.1.2 Analysis

Define the net saving rate by

$$s_t = \frac{a_{t+1}}{(1-\kappa)w_t(1-\tau_t^p)},$$

and note that with this definition of the saving rate we obtain the law of motion for capital in general equilibrium as

$$k_{t+1} = a_{t+1} = s_t(1-\alpha)k_t^\alpha(1-\kappa)(1-\tau_t^p)$$

and can thus express consumption in the two periods in general equilibrium as

$$
\begin{aligned}
c_t^y &= (1-s_t)(1-\tau_t^p)(1-\alpha)(1-\kappa)k_t^\alpha = \frac{1-s_t}{s_t}k_{t+1} \\
c_{t+1}^o &= k_{t+1}\alpha k_{t+1}^{\alpha-1} + k_{t+1}^\alpha(1-\alpha)\left(\kappa\eta + \tau_{t+1}^p(1-\kappa)\right) \\
&= \left(\alpha + (1-\alpha)\left(\kappa\eta + \tau_{t+1}^p\left(1-\kappa\right)\right)\right)k_{t+1}^\alpha.
\end{aligned}
$$

Using this in the private household Euler equation in competitive equilibrium with log utility

$$
1 = \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int \frac{c_t^y}{c_{t+1}^o(\eta)}d\Psi(\eta)
$$

yields

$$
\begin{aligned}
1 &= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int \frac{\frac{1-s_t}{s_t}k_{t+1}}{\left(\alpha + (1-\alpha)\left(\kappa\eta + \tau_{t+1}^p\left(1-\kappa\right)\right)\right)k_{t+1}^\alpha}d\Psi(\eta) \\
&= \alpha\beta(1-\tau_{t+1})\Gamma(\alpha,\kappa,\Psi;\tau_{t+1}^p)\frac{1-s_t}{s_t},
\end{aligned}
$$

where the constant summarizing the impact of income risk is now given by

$$
\Gamma(\alpha,\kappa,\Psi;\tau_{t+1}^p) = \int \left[\alpha + (1-\alpha)\left(\kappa\eta + \tau_{t+1}^p\left(1-\kappa\right)\right)\right]^{-1}d\Psi(\eta) \tag{45}
$$

and thus the private saving rate is only a function of the two tax rates and exogenous parameters:

$$
s_t(\tau_{t+1},\tau_{t+1}^p) = \frac{1}{1+\left[\alpha\beta(1-\tau_{t+1})\Gamma(\alpha,\kappa,\Psi;\tau_{t+1}^p)\right]^{-1}}. \tag{46}
$$

From equation (46) the following observation immediately follows:

Observation 1. *The private saving rate in general equilibrium with a PAYG pension system has the following properties:*

1. $\frac{\partial\Gamma(\cdot;\tau_{t+1}^p)}{\partial\tau_{t+1}^p} > 0$ *and thus* $\frac{\partial s(\tau_{t+1},\tau_{t+1}^p)}{\partial\tau_{t+1}^p} < 0.$
2. $\frac{\partial s(\tau_{t+1},\tau_{t+1}^p)}{\partial\tau_{t+1}} < 0$
3. *A mean-preserving spread in* η *increases* $\Gamma(\cdot;\tau_{t+1}^p)$ *and thus* $s(\tau_{t+1},\tau_{t+1}^p)$ *by less the*

larger is τ_{t+1}^{p}.

The key implication of this result is that for given τ_{t+1}^{p} we can implement any desired saving rate s_t by choice of τ_{t+1}. The saving rate increases in income risk, but less so with a larger pension system since the latter provides partial consumption insurance in old age, and thus reduces the precautionary saving incentives of private households.

Thus the implementation results from the main paper extend unchanged to the case with a PAYG pension system. If, in addition, the constraint $\tau_{t+1}^{p} \geq 0$ is imposed and is binding, then the implementation result from the main text applies unchanged (since the PAYG system is not operative in that case).

G.1.3 The Ramsey Tax Problem

From the implementation result we observe that any saving rate $s_t \in (0,1)$ can be implemented for a given contribution rate τ_t^p with some capital tax rate $\tau_t \in (-\infty, 1)$. In light of this, we define the Ramsey problem as one of directly choosing the saving rate s_t and the contribution rate to the pension system τ_t^p, which constitutes a hybrid between a primal and an indirect utility approach to optimal taxation.

The government's social welfare function is

$$W = \omega_{-1}\beta \int \ln(c_0^o(\eta_0))d\Psi(\eta_0) + \sum_{t=0}^{\infty} \omega_t \left[\ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1}))d\Psi(\eta_{t+1})\right] \quad (47)$$

which using the expressions

$$\begin{aligned} c_t^y &= (1-s_t)(1-\tau_t^p)(1-\alpha)(1-\kappa)k_t^{\alpha} \\ c_{t+1}^o &= \left(\alpha + (1-\alpha)\left(\kappa\eta + \tau_{t+1}^p(1-\kappa)\right)\right)k_{t+1}^{\alpha} \\ k_{t+1} &= s_t(1-\tau_t^p)(1-\kappa)(1-\alpha)k_t^{\alpha} \end{aligned}$$

can be rewritten as

$$
\begin{aligned}
W = \Xi + \beta\omega_{-1} \int \ln\left[\left(\alpha + (1-\alpha)\left(\kappa\eta_0 + \tau_0^p(1-\kappa)\right)\right)\right] d\Psi(\eta_0) \\
+ \sum_{t=0}^{\infty} \omega_t \left[\ln(1-s_t) + \ln(1-\tau_t^p) + \alpha\ln(k_t)\right] \\
+ \beta\sum_{t=0}^{\infty}\omega_t\left[\int \ln\left[\left(\alpha + (1-\alpha)\left(\kappa\eta_{t+1} + \tau_{t+1}^p(1-\kappa)\right)\right)\right] d\Psi(\eta_{t+1})\right. \\
\left. +\alpha\ln(1-\tau_t^p) + \alpha\ln(s_t) + \alpha^2\ln(k_t)\right].
\end{aligned}
$$

Now follow the analogous steps to those in Appendix B to write the dynamics of the capital stock as

$$
\ln(k_{t+1}) = \varkappa_{t+1} + \sum_{j=0}^{t-1}\alpha^j\left(\ln(s_{t-j}) + \ln\left(1-\tau_{t-j}^p\right)\right)
$$

and collect terms to get

$$
\begin{aligned}
W = \tilde{\Xi} + \beta\sum_{t=0}^{\infty}\omega_{t-1}\int \ln\left[\left(\alpha + (1-\alpha)\left(\kappa\eta_0 + \tau_t^p(1-\kappa)\right)\right)\right] d\Psi(\eta_t) \\
+ \sum_{t=0}^{\infty}\omega_t\left[\ln(1-s_t) + (\ln(s_t) + \ln(1-\tau_t^p))\left(\alpha\beta + \alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)\right].
\end{aligned}
$$

We directly observe from the above that the first-order condition with respect to s_t is the same as derived in Appendix B and therefore

$$
s_t^* = \frac{1}{1 + \left(\alpha\beta + \alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}}. \tag{48}
$$

For τ_t^p we obtain from the respective first-order condition in all $t = 0, \ldots$ the function

$$
f(\tau_t^p) = \beta\Gamma(\tau_t^{p^*})(1-\alpha)(1-\kappa) - \frac{1}{1-\tau_t^p}\frac{\omega_t}{\omega_{t-1}}\left(\alpha\beta + \alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right) \tag{49}
$$

which since $\Gamma(\tau_t^p) > 0$ is increasing in τ_t^p and since $\frac{1}{1-\tau_t^p}$ is convex and upward sloping in τ_t^p with $\lim_{\tau_t^p\to 1}\frac{1}{1-\tau_t^p} = \infty$ and $\lim_{\tau_t^p\to -\infty}\frac{1}{1-\tau_t^p} = 0$ implicitly defines the optimal $\tau_t^{p^*} \in$

$(-\infty, 1)$. Furthermore, since $\Gamma(\tau_t^p)$ is increasing in risk for a given τ_t^p we find that τ^{p^*} is increasing in risk (but need not be positive).

Furthermore, we know from (46), evaluated at the optimal saving rate s_t^* that the optimal capital tax rate required to implement s_t^* is given by

$$\tau_{t+1}(s_t^*, \tau_{t+1}^{p^*}) = 1 - \frac{s_t^*}{\alpha\beta(1-s_t^*)\Gamma(\tau_{t+1}^{p^*})}. \tag{50}$$

Holding the contribution rate to the pension system constant, the optimal τ_{t+1}^* is increasing in risk. However, to characterize the complete response of the capital tax rate to income risk we have to take into account that the contribution rate to the pension system also rises, reducing overall second period income risk and thus precautionary saving (and therefore the need to tax capital income). Introducing the notation that an increase of risk is measured by an increase of the variance σ_η^2 of η we therefore must evaluate the total derivative:[11]

$$\frac{\partial \tau_{t+1}}{\partial \sigma_\eta^2} = \frac{s_t^*}{\alpha\beta(1-s_t^*)\Gamma(\tau_{t+1}^{p^*})^2} \left(\underbrace{\frac{\partial \Gamma(\tau_{t+1}^p, \Psi)}{\partial \sigma_\eta^2}}_{>0} + \underbrace{\frac{\partial \Gamma(\tau_{t+1}^p, \Psi)}{\partial \tau_{t+1}^p}}_{<0} \underbrace{\frac{\partial \tau_{t+1}^p(\Psi)}{\partial \sigma_\eta^2}}_{>0} \right). \tag{51}$$

We next show that $\frac{\partial \tau_{t+1}}{\partial \sigma_\eta^2} > 0$. From the first-order condition (49) we note that by the implicit function theorem $\frac{d\tau_{t+1}^p}{d\sigma_\eta^2} = -\frac{\partial f(\cdot)/\partial \sigma_\eta^2}{\partial f(\cdot)/\partial \tau_{t+1}^p}$ with the partial derivatives

$$\frac{\partial f(\cdot)}{\partial \sigma_\eta^2} = \beta(1-\alpha)(1-\kappa)\frac{\partial \Gamma(\tau_{t+1}^p)}{\partial \sigma^2} > 0$$

$$\frac{\partial f(\cdot)}{\partial \tau_{t+1}^p} = -\left(\frac{\omega_t}{\omega_{t-1}} \left(\alpha\beta + \alpha\left(1+\alpha\beta\right) \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} \right) \frac{1}{(1-\tau^p)^2} - \beta(1-\alpha)(1-\kappa) \frac{\partial \Gamma(\tau_{t+1}^p)}{\partial \tau_{t+1}^p} \right) < 0.$$

We thus find (as we had already argued informally above) that the optimal social security

[11] In Appendix K.2 we show that the notion of an increase in income risk (mean-preserving spread in the distribution of η) is equivalent to an increase in the variance of η to a second order approximation of the integral Γ. Here expressing income risk in terms of the variance is simply a matter of notation, and stands in for a mean-preserving spread in η.

contribution rate is strictly increasing in income risk:

$$\frac{d\tau^p_{t+1}}{d\sigma^2_\eta} = \frac{\partial\Gamma/\partial\sigma^2_\eta}{\frac{\omega_t}{\omega_{t-1}}\frac{\alpha\beta+\alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}}{\beta(1-\kappa)(1-\alpha)}\frac{1}{(1-\tau^p)^2} - \partial\Gamma(\tau^p_{t+1})/\partial\tau^p_{t+1}} > 0$$

We now use this result to sign the overall effect of income risk on the optimal capital tax rate. For this, note that

$$\frac{\partial\Gamma(\tau^p_{t+1},\Psi)}{\partial\tau^p_{t+1}}\frac{\partial\tau^p_{t+1}(\Psi)}{\partial\sigma^2_\eta} = \frac{\partial\Gamma/\partial\sigma^2_\eta}{\frac{\omega_t}{\omega_{t-1}}\frac{\alpha\beta+\alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}}{\beta(1-\kappa)(1-\alpha)}\frac{1}{(1-\tau^p)^2}\frac{1}{\partial\Gamma(\tau^p_{t+1})/\partial\tau^p_{t+1}} - 1}$$

and we can rewrite (51) as

$$\frac{\partial\tau}{\partial\sigma^2_\eta} = \frac{s^*_t}{\alpha\beta(1-s^*_t)\Gamma(\tau^{p^*}_{t+1})^2}\frac{\partial\Gamma(\tau^p_{t+1},\Psi)}{\partial\sigma^2_\eta}\cdot \left(1 - \frac{1}{1-\frac{\omega_t}{\omega_{t-1}}\frac{\left(\alpha\beta+\alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)}{\beta(1-\kappa)(1-\alpha)}\frac{1}{(1-\tau^p)^2}\frac{1}{\partial\Gamma(\tau^p_{t+1})/\partial\tau^p_{t+1}}}\right)$$

and since $\frac{\partial\Gamma}{\partial\tau^p} < 0$ we obtain

$$\begin{aligned} & 1 - \frac{\omega_t}{\omega_{t-1}}\frac{\left(\alpha\beta+\alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)}{\beta(1-\kappa)(1-\alpha)}\frac{1}{(1-\tau^p)^2}\frac{1}{\partial\Gamma(\tau^p_{t+1})/\partial\tau^p_{t+1}} > 1 \\ \Leftrightarrow\quad & 1 - \frac{1}{1-\frac{\omega_t}{\omega_{t-1}}\frac{\left(\alpha\beta+\alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)}{\beta(1-\kappa)(1-\alpha)}\frac{1}{(1-\tau^p)^2}\frac{1}{\partial\Gamma(\tau^p_{t+1})/\partial\tau^p_{t+1}}} > 0 \\ \Leftrightarrow\quad & \frac{\partial\tau}{\partial\sigma^2_\eta} > 0. \end{aligned}$$

Therefore, the direct effect of a marginal increase of income risk on households savings in competitive equilibrium dominates the indirect effect from a reduction of consumption risk due to an increase of the optimal social security contribution and benefit system. The mitigation of the additional income risk through a marginal increase of the social security contribution rate τ^p_{t+1} is not strong enough to offset the effect of the marginal increase of risk. Intuitively, the Ramsey government, when optimally determining the social security contribution rate, has two motives. First, it aims at inter-generational redistribution. Second, it aims at reducing the direct effect of income risk. Since it has these two motives, it will not be optimal for the Ramsey government to completely offset a marginal increase of income risk so that $\Gamma(\tau^p_{t+1}; \Psi)$ increases even after the optimal adjustment of τ^p_{t+1}. Since the household saving rate in competitive equilibrium therefore increases due to the precautionary saving motive and since the Ramsey government aims at implementing a constant saving rate in order to offset the negative pecuniary externality from that increase of risk—just as in our model from the main text—the capital tax rate has to increase with income risk in order to implement that constant saving rate.

Finally, notice that with geometric discounting of the government $\omega_t = \theta^t$ we get

$$\sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} = \frac{\theta}{1-\alpha\theta}$$

and thus $\tau_t^{p^*} = \tau^{p^*}$, $s_t^* = s^*$ and $\tau_{t+1}^* = \tau^*$ for all t. We have the following:

Proposition 8. *The optimal saving rate s_t^* is independent of risk and the optimal pension contribution rate $\tau_{t+1}^{p^*}$ and the optimal capital tax rate τ_{t+1}^* are strictly increasing in idiosyncratic income risk. If, in addition, $\omega_t = \theta^t$ then $s_t^* = \frac{\alpha(\beta+\theta)}{1+\alpha\beta}$, $\tau_t^{p^*} = \tau^{p^*}$ and $\tau_{t+1}^* = \tau^*$ for all $t = 0, \ldots, \infty$.*

G.1.4 Ramsey Tax Problem in Steady State

We next aim to relate results in the pension system to those stated in Proposition 4 and thus from now on focus on maximizing steady state utility where $\omega_t = \theta = 1$. First, consider the deterministic economy. In this case the optimal pension contribution rate solves

$$\frac{1+\alpha\beta}{1-\alpha}\frac{1}{1-\tau^p} = \frac{\beta(1-\alpha)(1-\kappa)}{\alpha + (1-\alpha)\left(\kappa + \tau^p(1-\kappa)\right)}. \tag{52}$$

Solving this equation for the optimal contribution rate under certainty (denoted by $\Psi = \bar{\Psi}$) delivers

$$\tau^{p^*}(\bar{\Psi}) = \frac{\frac{\beta}{1+\beta} - \frac{\alpha}{1-\alpha}}{1-\kappa} - \frac{\kappa}{1-\kappa}. \tag{53}$$

The optimal capital tax rate is determined from our implementation result, equation (46):

$$\frac{1}{\alpha\beta\Gamma(\tau^{p^*}(\bar{\Psi}),\bar{\Psi})}\frac{s^*}{1-s^*} = 1-\tau^*(\bar{\Psi}). \tag{54}$$

We have

$$\frac{1}{\Gamma(\tau^{p^*}(\bar{\Psi}),\bar{\Psi})} = \alpha + (1-\alpha)\left(\kappa + (1-\kappa)\tau^{p^*}(\bar{\Psi})\right) = \frac{(1-\alpha)\beta}{1+\beta}.$$

Using this result and the expression for s^* in (54) we obtain for the optimal tax rate on capital $\tau^*(\bar{\Psi}) = 0$. The associated steady state capital stock is

$$k^*(\bar{\Psi}) = \left(s^*(1-\tau^{p^*}(\bar{\Psi}))(1-\alpha)(1-\kappa)\right)^{\frac{1}{1-\alpha}}.$$

Using the expressions for s^* and $\tau^{p^*}(\bar{\Psi})$ in the above expressions immediately implies that $s^* \cdot (1-\tau^{p^*}\bar{\Psi}) = s^{GR}$ and $k^*(\bar{\Psi}) = k^{GR}$. Furthermore, recall from Corollary 1 of Appendix E that for $\theta = 1$ and for a Pareto weight of $\omega_{-1} = 1$ on the initial old generation, the social planner maximizing steady state utility implements the saving rate s^{GR} along all periods of the transition. Therefore the Ramsey government setting s^* and $\tau^{p^*}(\bar{\Psi})$ from period 0 onward implements the socially optimal allocation. We summarize these results in the next

Proposition 9. *In the deterministic economy, for $\theta = 1$ and $\omega_{-1} = 1$, setting the optimal tax rates maximizing steady state utility*

$$\tau^{p^*}(\bar{\Psi}) = \frac{\frac{\beta}{1+\beta} - \frac{\alpha}{1-\alpha}}{1-\kappa} - \frac{\kappa}{1-\kappa} \;\; \textit{and} \;\; \tau^*(\bar{\Psi}) = 0 \tag{55}$$

in period 0 and holding them constant induces a transition path that implements the social optimum, with the golden rule saving rate $s^{GR} = \frac{\alpha}{(1-\kappa)(1-\alpha)}$. The economy converges monotonically to the golden rule steady state capital stock $k^* = k^{GR} = \alpha^{\frac{1}{1-\alpha}}$.

We also observe from the above that if the initial deterministic laissez-faire economy has a capital stock below the golden rule, $k_0^{CE} < k^{GR}$, then the optimal long-run steady state welfare maximizing social security contribution rate is negative. In contrast, if the competitive equilibrium capital stock is above the golden rule, $k_0^{CE} > k^{GR}$, then the optimal contribution rate is positive.

Since in the deterministic economy the Ramsey government implements the golden rule capital stock, since the optimal Ramsey net saving rate s^* is independent of income risk and since the optimal contribution rate τ^{p^*} to the pension system is strictly increasing in income risk we have the following

Corollary 4. *In the economy where η is risky, the optimal Ramsey long run capital stock satisfies $k^* < k^{GR}$.*

Related to Proposition 4 in the main text, we now establish that there is a threshold risk level such that for risk above that threshold we have $\tau^{p^*} > 0$, and for risk below the threshold, $\tau^{p^*} < 0$. That threshold lies in the intermediate risk range of Proposition 4.

To see this, recall that we have established that without income risk $\tau^{p^*}(\bar{\Psi})$ implements the golden rule capital stock, and the associated optimal tax on capital is $\tau^* = 0$. Since by the assumption on parameters maintained in Proposition 4 the laissez-faire competitive equilibrium capital stock is below the golden rule, $k(\bar{\Psi}) < k^{GR}$, implementing the golden rule capital stock without income risk requires $\tau^{p^*}(\bar{\Psi}) < 0$. Thus the starting point is the economy without risk and with optimal policy $\tau^{p^*} < 0, \tau^* = 0$. Now increase income risk.

We have established above that both τ^{p^*} and τ^* are strictly increasing in income risk. Thus there exists some threshold risk level $\hat{\Gamma}$ for which $\tau^{p^*} = 0$. Recall that the first order condition for τ^p (see equation (49)) is

$$\frac{1+\alpha\beta}{1-\alpha}\frac{1}{1-\tau^p} = \beta(1-\alpha)(1-\kappa)\Gamma(\tau^p).$$

which for $\tau^{p^*} = 0$ defines the risk threshold $\hat{\Gamma}$ explicitly as

$$\hat{\Gamma} = \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)} \tag{56}$$

To show that this threshold $\hat{\Gamma}$ lies in the intermediate risk interval of Proposition 4,

$\left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1}{\beta\left(1-\alpha-\frac{1}{\Gamma}\right)}\right)$, first investigate the lower bound of the interval. Notice that

$$
\begin{aligned}
& \hat{\Gamma} > \frac{1+\beta}{(1-\alpha)\beta} \\
\Leftrightarrow \quad & \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)} > \frac{1+\beta}{(1-\alpha)\beta} \\
\Leftrightarrow \quad & \frac{1+\alpha\beta}{(1-\alpha)(1-\kappa)} > 1+\beta \\
\Leftrightarrow \quad & s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)} > \frac{\alpha(1+\beta)}{1+\alpha\beta} = s^*
\end{aligned}
$$

and note that s^{GR} is defined as a gross saving rate whereas s^* is defined as a net saving rate. However, for $\tau^{p^*} = 0$, the gross and the net saving rates are identical. The inequality above follows from the proof of proposition 4: For the intermediate risk case we established there that $s^* < s^{GR}$, a result which carries over to the current analysis of social security as long as $\tau^{p^*} = 0$.

Now consider the upper bound. Notice that

$$
\begin{aligned}
& \hat{\Gamma} < \frac{1}{\beta\left(1-\alpha-\frac{1}{\Gamma}\right)} \\
\Leftrightarrow \quad & \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)} < \frac{1}{\beta\left(1-\alpha-(\kappa(1-\alpha)+\alpha)\right)} \\
\Leftrightarrow \quad & \frac{1+\alpha\beta}{(1-\alpha)^2(1-\kappa)} < \frac{1}{(1-\alpha)(1-\kappa)-\alpha} \\
\Leftrightarrow \quad & \frac{1}{(1-\alpha)(1-\kappa)} < \frac{1}{(1+\alpha\beta)\left((1-\kappa)-\frac{\alpha}{1-\alpha}\right)},
\end{aligned}
$$

and since $s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)} < 1 \;\Leftrightarrow\; (1-\kappa) - \frac{\alpha}{1-\alpha} > 0$ we further get

$$
\begin{aligned}
\Leftrightarrow \qquad & (1-\alpha)(1-\kappa) > (1+\alpha\beta)\left((1-\kappa) - \frac{\alpha}{1-\alpha}\right) \\
\Leftrightarrow \qquad & (1-\alpha)(1-\kappa) > (1+\alpha\beta)(1-\kappa) - \frac{\alpha(1+\alpha\beta)}{1-\alpha} \\
\Leftrightarrow \qquad & (1-\kappa) - \alpha(1-\kappa) > (1-\kappa) + \alpha\beta(1-\kappa) - \frac{\alpha(1+\alpha\beta)}{1-\alpha} \\
\Leftrightarrow \qquad & \frac{\alpha(1+\alpha\beta)}{1-\alpha} > \alpha\beta(1-\kappa) + \alpha(1-\kappa) \\
\Leftrightarrow \qquad & s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)} > \frac{\alpha(1+\beta)}{(1+\alpha\beta)} = s^*
\end{aligned}
$$

and again the above inequality follows from proposition 4. Therefore the threshold satisfies $\hat{\Gamma} \in \left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1}{\beta\left(1-\alpha-\frac{1}{\Gamma}\right)}\right)$, that is, lies in the intermediate risk interval of proposition 4 in the main text. With this characterization of $\hat{\Gamma}$ we can state the next proposition, which serves as a generalization of proposition 4. It characterizes the jointly optimal pension contribution and capital tax rate, and also covers the case when a nonnegativity constraint on pension contributions and thus pension benefits is imposed.

Proposition 10. *Let $\theta = 1$ so that the Ramsey government maximizes steady state welfare. Denote by s^{CE} the saving rate in the laissez-faire competitive equilibrium and by s^{GR} the gross saving rate that implements the golden rule capital stock. Further denote by s^* the optimal Ramsey net saving rate, $s^* = \frac{a^*}{w(1-\tau^{p^*})}$, where τ^{p^*} is the optimal Ramsey pension contribution rate. Finally denote by τ^* the optimal Ramsey capital tax rate.*

1. *Let income risk be **large**, $\Gamma > \frac{1}{\beta\left((1-\alpha)-\frac{1}{\Gamma}\right)}$. Then $s^{CE} > s^{GR} > s^*$, and $\tau^* > 0$, and $\tau^{p^*} > 0$.*
2. *Let income risk be **fairly large**, $\Gamma \in \left(\frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}, \frac{1}{\beta\left((1-\alpha)-\frac{1}{\Gamma}\right)}\right)$. Then $s^{CE} < s^{GR}$ and $\tau^* > 0$, and $\tau^{p^*} > 0$, and thus $s^* < s^{CE}$.*
3. *Let income risk be **fairly small**, $\Gamma \in \left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}\right)$. Then $s^{CE} < s^{GR}$ and $\tau^* > 0$. If the social security contribution rate is unrestricted, then $\tau^{p^*} < 0$. If it is subject to a nonnegativity constraint, then $\tau^{p^*} = 0$ and $s^* < s^{CE}$.*
4. *Let income risk be **small**, $\Gamma \in \left(\bar{\Gamma}, \frac{1+\beta}{(1-\alpha)\beta}\right)$. Then $s^{CE} < s^{GR}$. If the social security contribution rate is unrestricted, then $\tau^* > 0$ and $\tau^{p^*} < 0$. If it is subject to a*

nonnegativity constraint, then $\tau^* < 0, \tau^{p^*} = 0$, and $s^* < s^{CE}$.

The interesting interval is thus the interval where risk is *fairly small*, where the optimal capital tax is positive but the pension contribution rate is negative (or zero, if constrained to be nonnegative). To provide some intuition for this finding, notice that the optimal pension contribution rate turns positive in the stochastic economy at a level of risk that is below the risk level where the laissez-faire competitive equilibrium economy's capital stock is equal to the golden rule capital stock because the pension system serves two purposes: it provides optimal intergenerational redistribution and it partially insures against idiosyncratic income risk. This dual role can directly be inferred from the first-order condition (49). Given the contribution rate and the implied remaining idiosyncratic consumption risk (which cannot be fully eliminated by social security benefits), the capital income tax implements the optimal Ramsey saving rate, offsetting the negative pecuniary externality from increasing saving rates induced by income risk of households, exactly as in the model without social security. If the pension contribution rate is restricted to zero, then the tax on capital also targets inter-generational redistribution, as in the benchmark model. So why does the optimal tax rate on capital turn positive for a lower threshold of risk with, compared to without social security? Without social security as an inter-generational redistribution instrument, capital taxation partially fills the role of providing inter-generational redistribution in addition to addressing the pecuniary externality, and in the case the competitive equilibrium capital stock is below the golden rule, this force pushes down the tax on capital (to encourage capital accumulation) relative to the case where social security tackles the desired intergenerational redistribution (through a negative contribution rate, if permitted).

Finally, denote by $k^*(\tau^*, \tau^p = 0)$ the optimal Ramsey steady state capital stock in the economy without a pension system and by $k^*(\tau^*, \tau^{p^*})$ the steady state capital stock in the economy with a pension system. From the optimal saving rate and the optimal pension contribution rate characterized in Proposition 10 we obtain the next

Corollary 5. *The optimal long run capital stock in the economy with and the economy without social security are related as follows:*

1. *For large income risk, $\Gamma > \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}$, we have $k^*(\tau^*, \tau^{p^*}) < k^*(\tau^*, \tau^p = 0)$*
2. *For small income risk, $\Gamma \leq \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}$, we have $k^*(\tau^*, \tau^{p^*}) \geq k^*(\tau^*, \tau^p = 0)$*

This result immediately follows from the fact that the optimal net savings rates satisfy $s^*(\tau^*, \tau^p = 0) = s^*(\tau^*, \tau^{p^*})$ and the steady state capital stock follows from the saving rate as $k^* = (s^*(1-\tau^p)(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}$. For $\Gamma > \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}$, proposition 10 implies that $\tau^{p^*} > 0$ (and thus $k^*(\tau^*, \tau^{p^*}) < k^*(\tau^*, \tau^p = 0)$), and for $\Gamma < \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}$ we have $\tau^{p^*} \leq 0$ (and thus $k^*(\tau^*, \tau^{p^*}) \geq k^*(\tau^*, \tau^p = 0)$) where the equality is strict if the constraint $\tau^{p^*} \geq 0$ applies.

Transition Under Optimal Steady State Policy Under the optimal long-run steady state welfare maximizing policy τ^*, τ^{p^*} implemented in period 0, the economy converges to the long steady state with the dynamics of the capital stock along the transition given by

$$k_{t+1} = s^*(1-\tau^{p^*})(1-\kappa)(1-\alpha)k_t^\alpha$$

and transfers to the initial old generation of

$$b_0 = \tau^{p^*}(1-\kappa)(1-\alpha)k_0^\alpha$$

so that initial consumption of the old is

$$\begin{aligned} c_0^o &= a_0 R_0 + \kappa\eta w_0 + b_0 \\ &= k_0 \alpha k_0^{\alpha-1} + \kappa\eta(1-\alpha)k_0^\alpha + \tau^{p^*}(1-\kappa)(1-\alpha)k_0^\alpha \\ &= \left(\alpha + (1-\alpha)\left(\kappa\eta + \tau^{p^*}(1-\kappa)\right)\right)k_0^\alpha. \end{aligned}$$

G.1.5 Illustration

Panels (a) and (b) of Figure 1 illustrate Proposition 10 by plotting the optimal social security contribution rate and the optimal capital tax rate against the extent of income risk. It does so both for the case in which there is a nonnegativity constraint on τ^p and the case where the social security contribution rate is unconstrained. Panels (c) and (d) in the figure show the policy instruments for optimal government debt to which we turn in Section G.2.

Panels (d) and (b) first demonstrate that both the social security contribution rate and the capital tax rate are increasing in income risk, strictly so for τ^* and also for τ^{p^*} unless the latter is constrained to be nonnegative, in which case $\tau^{p^*} = 0$ if income risk is small, see Panel (b). The vertical lines separate the x-axis into the four intervals characterized in proposition 10. In the last two intervals (income risk fairly large and large), both tax rates are positive, strictly increasing in risk and the nonnegativity constraint on τ^p is not binding. Below the threshold associated with $\hat{\Gamma}$ the payroll tax is either constrained at zero and the tax on capital is rising in income risk and turns from negative to positive at the first threshold characterized in the original proposition 4. Alternatively, there is no constraint, in which case τ^* is unambiguously positive and rising in income risk, and τ^{p^*} is also strictly increasing in income risk but negative for small and fairly small (in the nomenclature of proposition 10) income risk. Finally, we observe that the impact of increased income risk on the tax rate on capital is smaller when the social security contribution rate is free to adjust (i.e. can be negative) than when it is constrained to be nonnegative, see the respective slopes of the two lines.

G.2 Equivalence of Social Security and Government Debt

In this subsection we establish equivalence of the optimal Ramsey allocations for a general social discount function $\boldsymbol{\omega}$ when the government has assess to a PAYG social security system analyzed in the previous section, and when, alternatively, it has access to government debt. We first characterize the policy instruments and allocations in the economy with debt and subsequently prove the equivalence by showing that a given allocation implemented by the pension-taxation policy can be implemented by the debt-taxation policy and vice-versa, and by arguing that the solution to the Ramsey maximization problem is unique.

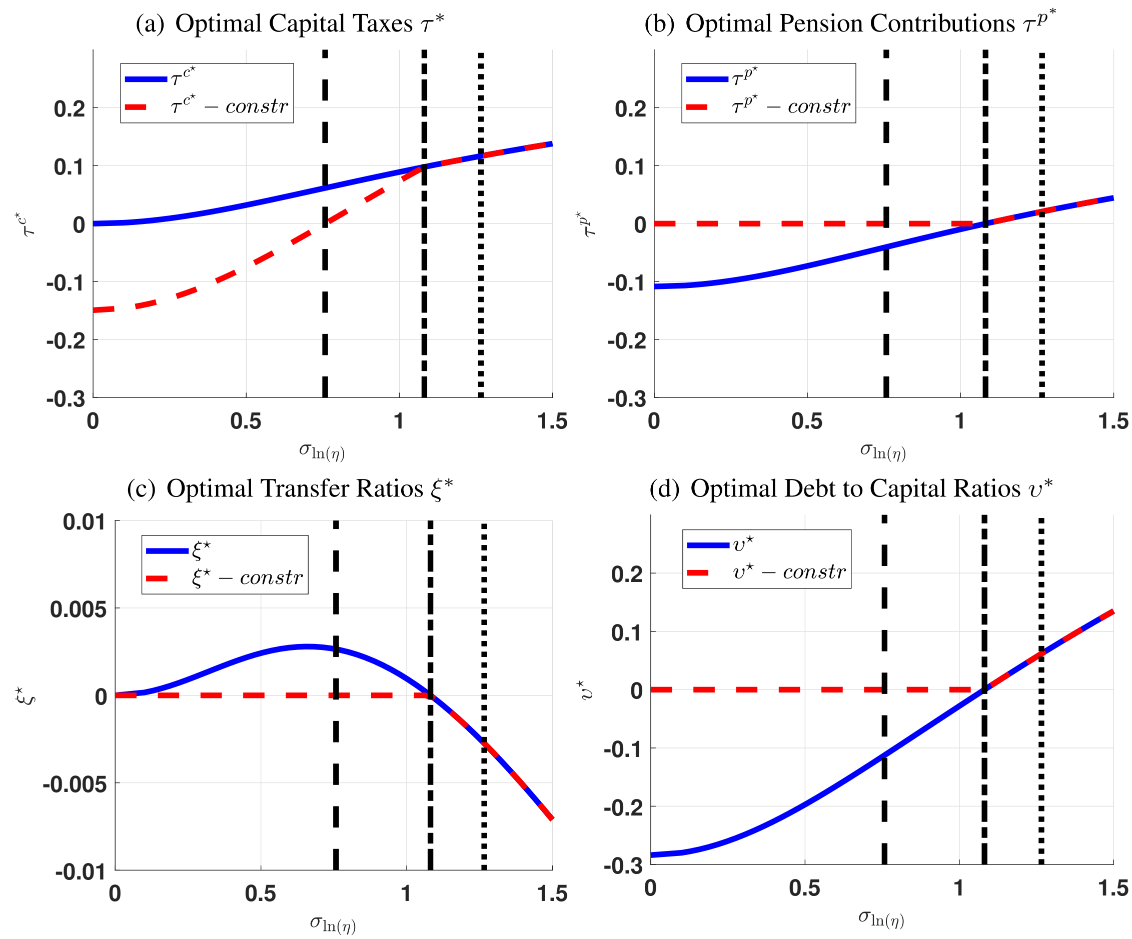
G.2.1 The Economy with Government Debt

In period 0 the initial government debt position in the laissez-faire competitive equilibrium is assumed to be $b_0 = 0$. The government then pays transfers Z_0 to the initial old households and finances these transfers by issuing government debt b_1. In all other periods, the government finances transfers to the old of Z_t with government debt, in addition to the transfers T_t financed by capital income taxes. Thus, the government budget constraint is, in each period,

$$b_{t+1} = \begin{cases} Z_0 & \text{for } t = 0 \\ R_t b_t + T_t - \tau_t R_t a_t + Z_t = R_t b_t + Z_t & \text{for } t > 0, \end{cases}$$

where we maintain the assumption from the benchmark model that the government sets transfers T_t equal to revenues from capital taxation, $T_t = \tau_t R_t a_t$. The household budget

Figure 1: Optimal Tax on Capital, Optimal Pension Contributions & Optimal Debt Instruments



Notes: Panel (a) shows the optimal capital income tax τ^*, Panel (b) the optimal pension contribution rate τ^{p^*}, Panel (c) the optimal transfer to wage ratio ξ^* (on a different scale than the other panels), and Panel (d) the optimal debt to capital ratio υ^* as a function of income risk, measured by the standard deviation, $\sigma_{\ln \eta}$. Solid blue lines are for the unconstrained solution, red dashed lines for the constrained solution with $\tau^{p^*} \geq 0$, respectively $\upsilon^* \geq 0$. The vertical lines separate the four risk intervals characterized in Proposition 10.

constraints in both periods of life read as

$$c_t^y + a_{t+1} = (1 - \kappa)w_t$$

$$c_t^o = a_t R_t (1 - \tau_t) + \kappa \eta w_t + T_t + Z_t$$

and equilibrium in the asset market now requires that private assets equal the capital stock and the stock of outstanding government debt:

$$a_t = k_t + b_t.$$

G.2.2 Analysis

Define the gross saving rate as

$$\tilde{s}_t = \frac{a_{t+1}}{(1-\kappa)w_t}.$$

Also define the ratio of government debt to the capital stock as $\upsilon_t = \frac{b_t}{k_t}$. With this definition the law of motion of the capital stock in general equilibrium can be derived from

$$a_{t+1} = \tilde{s}_t(1-\alpha)(1-\kappa)k_t^{\alpha} = k_{t+1} + b_{t+1} = (1+\upsilon_{t+1})k_{t+1}$$

as

$$k_{t+1} = \frac{\tilde{s}_t}{1+\upsilon_{t+1}}(1-\alpha)(1-\kappa)k_t^{\alpha}. \tag{57}$$

Finally, define the transfer rate $\xi_t = \frac{Z_t}{(1-\kappa)(1-\alpha)k_t^{\alpha}}$ and use it in the government budget constraint to obtain an alternative representation of the law of motion of the capital stock:

$$\begin{aligned}
b_{t+1} &= b_t R_t + Z_t \\
&= b_t \alpha k_t^{\alpha-1} + \xi_t(1-\kappa)(1-\alpha)k_t^{\alpha} \\
\Leftrightarrow \qquad \upsilon_{t+1}k_{t+1} &= \left[\alpha\upsilon_t + \xi_t(1-\kappa)(1-\alpha)\right]k_t^{\alpha} \\
\Leftrightarrow \qquad k_{t+1} &= \left[\alpha\frac{\upsilon_t}{\upsilon_{t+1}} + \frac{\xi_t}{\upsilon_{t+1}}(1-\kappa)(1-\alpha)\right]k_t^{\alpha}. \qquad (58)
\end{aligned}$$

Comparing (57) to (58) gives the law of motion for the debt to capital ratio $\upsilon_{t+1} = \frac{b_{t+1}}{k_{t+1}}$ as a function of $\upsilon_t, \xi_t, \tilde{s}_t$:

$$\begin{aligned}
& \frac{\tilde{s}_t}{1+\upsilon_{t+1}}(1-\kappa)(1-\alpha) = \alpha\frac{\upsilon_t}{\upsilon_{t+1}} + \frac{\xi_t}{\upsilon_{t+1}}(1-\kappa)(1-\alpha) \\
\Leftrightarrow \quad & \frac{\upsilon_{t+1}}{1+\upsilon_{t+1}} = \frac{1}{\tilde{s}_t}\left(\xi_t + \frac{\alpha}{(1-\kappa)(1-\alpha)}\upsilon_t\right) \\
\Leftrightarrow \quad & \upsilon_{t+1} = \frac{\xi_t + \frac{\alpha}{(1-\kappa)(1-\alpha)}\upsilon_t}{\tilde{s}_t - \left(\xi_t + \frac{\alpha}{(1-\kappa)(1-\alpha)}\upsilon_t\right)}
\end{aligned} \tag{59}$$

Finally, turn to the solution of the household model. Using (57) we can rewrite consumption of young and old households as

$$\begin{aligned}
c_t^y &= (1-\tilde{s}_t)(1-\alpha)(1-\kappa)k_t^\alpha = \frac{1-\tilde{s}_t}{\tilde{s}_t}(1+\upsilon_{t+1})k_{t+1} \\
c_{t+1}^o &= \left[\alpha + (1-\alpha)\left(\kappa\eta + (1-\kappa)\left(\xi_{t+1} + \frac{\alpha\upsilon_{t+1}}{(1-\alpha)(1-\kappa)}\right)\right)\right]k_{t+1}^\alpha.
\end{aligned}$$

Using this in the competitive equilibrium household Euler equation with log utility

$$1 = \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int \frac{c_t^y}{c_{t+1}^o(\eta)}d\Psi(\eta)$$

yields

$$\begin{aligned}
1 &= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int \frac{\frac{1-\tilde{s}_t}{\tilde{s}_t}(1+\upsilon_{t+1})k_{t+1}}{\left(\alpha + (1-\alpha)\left(\kappa\eta + \left(\xi_{t+1} + \frac{\alpha\upsilon_{t+1}}{(1-\alpha)(1-\kappa)}\right)\right)\right)k_{t+1}^\alpha}d\Psi(\eta) \\
&= \alpha\beta(1-\tau_{t+1})(1+\upsilon_{t+1})\hat{\Gamma}(\alpha,\kappa,\Psi;\xi_{t+1},\upsilon_{t+1})\frac{1-\tilde{s}_t}{\tilde{s}_t},
\end{aligned}$$

where the constant summarizing the impact of income risk is now given by

$$\Gamma(\alpha,\kappa,\Psi;\xi_{t+1},\upsilon_{t+1}) = \int\left[\alpha + (1-\alpha)\left(\kappa\eta + \left(\xi_{t+1} + \frac{\alpha\upsilon_{t+1}}{(1-\alpha)(1-\kappa)}\right)\right)\right]^{-1} d\Psi(\eta) \tag{60}$$

and thus the private saving rate is a function of the capital tax rate, the transfer rate and the ratio of debt to the capital stock as well as exogenous parameters:

$$\tilde{s}_t(\tau_{t+1}, \upsilon_{t+1}) = \frac{1}{1 + [\alpha\beta(1-\tau_{t+1})(1+\upsilon_{t+1})\Gamma(\alpha, \kappa, \Psi; \xi_{t+1}, \upsilon_{t+1})]^{-1}}. \tag{61}$$

Therefore, the implementation result for the tax rate on capital now takes the following form: given the period t debt to capital ratio υ_t and period t choices of the government ξ_t and a saving rate $\tilde{s}_t$, we obtain υ_{t+1} from equation (59) and can thus compute the tax rate τ_{t+1} implementing the private saving in competitive equilibrium from equation (61).

G.2.3 Proof of Equivalence

We first establish that a given allocation implemented by policy instruments of a pension-taxation policy can equivalently be implemented with policy instruments of a debt-taxation policy. Likewise we show that a given allocation implemented by policy instruments of a debt-taxation policy can be implemented with policy instruments of a pension-taxation policy. Formally, this equivalence is stated in the next

Proposition 11.

1. *Consider an allocation $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$ implemented with a pension-taxation policy $\{\tau_t^p, \tau_t\}_{t=0}^{\infty}$ with associated saving rate $\{s_t\}_{t=0}^{\infty}$. The same allocation can be implemented by a debt-taxation policy $\{\xi_t, \tau_t\}_{t=0}^{\infty}$ with associated saving rate and debt to capital ratio $\{\tilde{s}_t, \upsilon_{t+1}\}_{t=0}^{\infty}$.*

2. *Consider an allocation $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$ implemented with a debt-taxation policy $\{\xi_t, \tau_t\}_{t=0}^{\infty}$ with associated saving rate and debt to capital ratio $\{\tilde{s}_t, \upsilon_{t+1}\}_{t=0}^{\infty}$. The same allocation can be implemented by a pension-taxation policy $\{\tau_t^p, \tau_t\}_{t=0}^{\infty}$ with associated saving rate $\{s_t\}_{t=0}^{\infty}$.*

Proof. Recall that the allocations and their dependency on policy instruments in the debt-taxation policy are given by

$$c_t^y = (1-\tilde{s}_t)(1-\alpha)(1-\kappa)k_t^{\alpha} \tag{62a}$$

$$c_t^o = \left[\alpha + (1-\alpha)\left(\kappa\eta + (1-\kappa)\left(\xi_t + \frac{\alpha\upsilon_t}{(1-\alpha)(1-\kappa)}\right)\right)\right] k_t^{\alpha} \tag{62b}$$

$$k_{t+1} = \frac{\tilde{s}_t}{1+\upsilon_{t+1}}(1-\alpha)(1-\kappa)k_t^{\alpha} \tag{62c}$$

whereas in the pension-taxation policy they are given by

$$c_t^y = (1 - s_t)(1 - \tau_t^p)(1 - \alpha)(1 - \kappa)k_t^\alpha \tag{63a}$$

$$c_t^o = (\alpha + (1 - \alpha)(\kappa\eta + (1 - \kappa)\tau_t^p))\, k_t^\alpha \tag{63b}$$

$$k_{t+1} = s_t(1 - \tau_t^p)(1 - \kappa)(1 - \alpha)k_t^\alpha. \tag{63c}$$

1. To establish part 1 of the proposition consider the following forward iteration in time from $t = 0, \ldots, \infty$, starting at $k_0, \upsilon_0 = 0$. In any period $t \geq 0$ for a given k_t, υ_t:

   (a) From (62b) and (63b) the consumption allocation of the period t-old implemented by a pension-taxation policy can be equivalently implemented by a debt-taxation policy through

   $$\xi_t = \tau_t^{p^*} - \frac{\alpha \upsilon_t^*}{(1 - \kappa)(1 - \alpha)} \tag{64}$$

   which for $t = 0$ gives $\xi_0 = \tau_0^p$.

   (b) From (62a) and (63a) the consumption allocation of the period t young implemented by a pension-taxation policy can be equivalently implemented by a debt-taxation policy through:

   $$\tilde{s}_t = s_t + \tau_t^p(1 - s_t). \tag{65}$$

   (c) Equivalence implies a path of government debt. In particular, this path can be inferred from the pension-taxation policy by using (64) in (59) to get

   $$\upsilon_{t+1} = \frac{1}{s_t\left(\frac{1}{\tau_t^p} - 1\right)}. \tag{66}$$

   (d) Finally, notice from the households' first-order condition that with substitutions (64)-(66) we obtain

$$
\begin{aligned}
1 &= \beta\alpha(1-\tau_{t+1})\frac{1-\tilde{s}_t}{\tilde{s}_t}(1+\upsilon_{t+1}) \\
&\quad \cdot \int \frac{1}{\alpha+(1-\alpha)\left(\kappa\eta+(1-\kappa)\left(\xi_{t+1}+\upsilon_{t+1}\frac{\alpha}{(1-\alpha)(1-\kappa)}\right)\right)} d\Psi(\eta) \\
&= \beta\alpha(1-\tau_{t+1})\frac{1-\tilde{s}_{t+1}}{\tilde{s}_{t+1}}(1+\upsilon_{t+1})\Gamma(\tau^p_{t+1}) \\
&= \beta\alpha(1-\tau_{t+1})\frac{1-s_{t+1}}{s_{t+1}}\Gamma(\tau^p_{t+1})
\end{aligned} \tag{67}
$$

2. To establish part 2 of the proposition we proceed analogously by inverting (65) to obtain τ^p_t, by inverting (64) to obtain s_t, and by using s_t, τ^p_{t+1} in (67) to obtain τ_{t+1}.

□

Finally, we can verify that the evolution of the capital stock is the same under both policies by comparing equations (62c) and (63c) to obtain

$$
\begin{aligned}
&\frac{\tilde{s}_t}{1+\upsilon_{t+1}} = s_t(1-\tau^p_t) \\
\Leftrightarrow \qquad &\upsilon_{t+1} = \frac{\tilde{s}_t}{s_t(1-\tau^p_t)} - 1 = \frac{s_t(1-\tau^p_t)+\tau^p_t}{s_t(1-\tau^p_t)} - 1 = \frac{\tau^p_t}{s_t(1-\tau^p_t)},
\end{aligned}
$$

which is the same as (66).

Therefore, we have thus shown that the same set of allocations can be implemented by either of the two policy instruments. Furthermore, since maximizing the strictly concave objective function (47) subject to either the linear constraints (62) or the linear constraints (63) are convex maximization problems, the respective solutions are unique. Thus, the optimal allocation implemented by the one policy (social security and capital taxes) can be implemented by the respective other policy (government debt and capital taxes). □

G.2.4 Characterization of the Optimal Ramsey Debt-Taxation Policy

We now want to characterize the optimal debt-taxation policy ξ^*_t, τ^*_t with associated optimal saving rate and optimal debt to capital ratio $\tilde{s}^*_t, \upsilon^*_{t+1}$. Observe from (65) and (66) that υ^*_{t+1} and $\tilde{s}^*_t$ are increasing in income risk, because $\tau^{p^*}_t$ is increasing in income risk and because s^*_t is constant in income risk. As a consequence, we see from (64) that it is ambiguous how ξ^*_t

varies with income risk. Also observe from (66) that $\upsilon^*_{t+1} \gtreqless 0$ if $\tau^{p^*}_t \gtreqless 0$ and therefore the qualitative behavior of debt is the same as of the pension payments.

Suppose next that the Ramsey government's discount function is geometric so that $\omega_t = \theta^t$ for some $\theta \in (0, 1]$. Notice from (65)–(67) that then the debt policy instruments are constant over time, $\xi^*_t = \xi^*, \tau^*_t = \tau^*, \upsilon^*_{t+1} = \upsilon^*$, because, as established above, the optimal contribution and saving rates in the pension system are constant, $\tau^{p^*}_t = \tau^{p^*}, s^*_t = s^*$.

Finally, consider a Ramsey government maximizing utility in steady state, hence $\theta = 1$. Under this assumption we now characterize how ξ^* varies with income risk. From the analysis of the deterministic economy above recall that

$$\frac{\alpha}{(1-\alpha)(1-\kappa)} = s^*(1 - \tau^{p^*}(\bar{\Psi}))$$

and next use (64) and (66) to rewrite ξ^* as

$$\xi^* = \tau^{p^*} \left(1 - \frac{1 - \tau^{p^*}(\bar{\Psi})}{1 - \tau^{p^*}}\right). \tag{68}$$

Expressing ξ^* in terms of τ^{p^*} leads us to the following cases concerning the dependence of optimal debt policy on income risk. First, consider the case that the deterministic laissez-faire competitive equilibrium has a capital stock below the golden rule. Then, at $\tau^{p^*}(\bar{\Psi}) < 0$ we have $\xi^* = 0$, and at $\hat{\Gamma}$ (see equation (56), we have $\tau^{p^*}(\hat{\Gamma}) = 0$ and thus $\xi^*(\hat{\Gamma}) = 0$. Furthermore, since for all $\Gamma \in (\bar{\Gamma}, \hat{\Gamma})$ $\tau^{p^*}(\bar{\Psi}) < \tau^{p^*} < 0$ we find that $\xi^* > 0$ for $\Gamma \in (\bar{\Gamma}, \hat{\Gamma})$, whereas for all $\Gamma > \hat{\Gamma}$ we have $\tau^{p^*}(\bar{\Psi}) < 0 < \tau^{p^*}$ and thus $\xi^* < 0$ for $\Gamma > \hat{\Gamma}$. Since in the deterministic economy $\xi^*(\bar{\Psi}) = 0$, the government finances some initial transfers to the period 0 old of Z_0 and then rolls over this debt into the future.

In the stochastic economy, however, the Ramsey government pays additional positive transfers to the period 0 old as long as risk is below the threshold level where social security turns positive and levies lump-sum taxes on the old for risk beyond that threshold. Second, in case the deterministic competitive equilibrium economy has a capital stock already above the golden rule, then for all $\Gamma > \bar{\Gamma}$ the optimal ξ^* is negative and falling in income risk.

G.2.5 Illustration

For a numerical illustration we return to Figure 1. Panels (c) and (d) of this figure show the optimal transfer to wage ratio ξ^* and the optimal debt to capital ratios υ^* as a function of income risk. As with social security, it does so both for the case in which there is a nonnegativity constraint on υ and the case where the debt level is unconstrained.

First turn to Panel (d) which illustrates that the optimal debt to capital ratio has the same properties as the optimal pension contribution rate. If the constraint $\upsilon \geq 0$ does not apply, the debt level is negative as long as income risk is small or fairly small, and it is increasing in risk, turning positive for large income risk. Finally, panel (c) shows the non-monotonicity of the transfer ratio (on a different scale than the other panels of the figure), which is positive if income risk is small and fairly small and (increasingly) negative for fairly large and large income risk.

G.3 Bequest Motive

In this section we provide the detailed analysis of the model with survival risk and warm-glow bequest motives. Assume now that households survive to the second period with probability $\varsigma \in (0, 1)$. In the second period of life they receive flow utility from own consumption in case of survival and from bequests, including interest net of taxes, in case of death.

G.3.1 Households

We assume that bequest utility takes the same functional form (log utility) as utility from consumption with utility weight parameter $\varphi > 0$. Accordingly the objective is

$$\max_{c_t^y, c_{t+1}^o, a_{t+1}^o} \ln(c_t^y) + \beta \mathbb{E}_t \left[\varsigma \ln(c_{t+1}^o) + (1-\varsigma)\varphi \ln\left(a_{t+1}^o R_{t+1}(1-\tau_{t+1})\right)\right].$$

and maximization is subject to the per period budget constraints

$$\begin{aligned} c_t^y + a_{t+1}^o &= (1-\kappa)w_t + a_t^y + T_t^y =: x_t \\ c_{t+1}^o &= a_{t+1}^o R_{t+1}(1-\tau_{t+1}) + \frac{\kappa}{\varsigma}\eta w_{t+1} + T_{t+1}^o \end{aligned}$$

where a_t^y are initial assets from warm-glow bequests and x_t is cash in hand of young households. We denote transfers from the government to the young and old by T_t^y, T_{t+1}^o, respectively. We further make the following

Assumption 3. *The total effective utility weight on bequests satisfies*

$$\beta(1-\varsigma)\varphi < 1.$$

G.3.2 Capital Market Equilibrium

Scaling of labor productivity in the second period by ς achieves that labor in the economy again aggregates to one

$$L_t = (1-\kappa) + \frac{\kappa}{\varsigma}\varsigma \int \eta d\Psi(\eta) = 1$$

so that $k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = K_{t+1}$ still applies. The capital market clearing condition reads as

$$K_{t+1} = k_{t+1} = a^o_{t+1} = s_t x_t.$$

where s_t is the private saving rate out of cash in hand.

G.3.3 Bequests

The aggregate amount of bequests distributed to the period $t+1$ young generation is

$$B_{t+1} = (1-\varsigma) a^o_{t+1} R_{t+1}(1-\tau_{t+1}).$$

Since the size of the young population is of measure 1 and they receive all bequests, initial assets of this generation are given by

$$a^y_{t+1} = (1-\varsigma) a^o_{t+1} R_{t+1}(1-\tau_{t+1}).$$

G.3.4 Government

Total government tax revenue is

$$T_{t+1} = a^o_{t+1} R_{t+1} \tau_{t+1}.$$

By assumption, tax revenues are redistributed to the young and old $T_{t+1} = T^y_{t+1} + \varsigma T^o_{t+1}$ according to the rule

$$\begin{aligned} T^y_{t+1} &= (1-\varsigma) a^o_{t+1} R_{t+1} \tau_{t+1} \\ \varsigma T^o_{t+1} &= \varsigma a^o_{t+1} R_{t+1} \tau_{t+1}. \end{aligned}$$

G.3.5 Household Maximization

Recall that cash in hand of young households is defined as $x_t = (1-\kappa)w_t + a_t^y + T_t^y$. Using the budget constraints we can rewrite the objective of a household born in t as

$$\max_{a_{t+1}^o} \ln(x_t - a_{t+1}^o) +$$
$$\beta\left(\mathbb{E}_t\left[\varsigma \ln\left(a_{t+1}^o R_{t+1}(1-\tau_{t+1}) + \frac{\kappa}{\varsigma}\eta w_{t+1} + T_{t+1}^o\right)\right] + (1-\varsigma)\varphi\left[\ln\left(a_{t+1}^o\right) + \ln\left(R_{t+1}(1-\tau_{t+1})\right)\right]\right)$$

and from the last term we observe that τ_{t+1} must be strictly less than one, $\tau_{t+1} < 1$, for the household maximization problem to be well-defined and a competitive equilibrium to exist. The first order condition with respect to a_{t+1}^o is given by

$$-\frac{1}{c_t^y} + \beta\left(\mathbb{E}_t\left[\varsigma\frac{1}{c_{t+1}^o}\right]R_{t+1}(1-\tau_{t+1}) + (1-\varsigma)\varphi\frac{1}{a_{t+1}^o}\right) = 0$$

and thus

$$\begin{aligned}
1 &= \beta\left(\mathbb{E}_t\left[\varsigma\frac{c_t^y R_{t+1}(1-\tau_{t+1})}{c_{t+1}^o}\right] + (1-\varsigma)\varphi\frac{c_t^y}{a_{t+1}^o}\right) \\
&= \beta\left(\mathbb{E}_t\left[\varsigma\frac{((1-s_t)x_t)\,R_{t+1}(1-\tau_{t+1})}{c_{t+1}^o}\right] + (1-\varsigma)\varphi\frac{(1-s_t)x_t}{s_t x_t}\right) \\
&= \beta\left(\mathbb{E}_t\left[\varsigma\frac{((1-s_t)x_t)\,R_{t+1}(1-\tau_{t+1})}{a_{t+1}^o R_{t+1}(1-\tau_{t+1}) + \frac{\kappa}{\varsigma}\eta w_{t+1} + T_{t+1}^o}\right] + (1-\varsigma)\varphi\frac{(1-s_t)}{s_t}\right) \\
&= \beta\left(\mathbb{E}_t\left[\varsigma\frac{((1-s_t)x_t)\,R_{t+1}(1-\tau_{t+1})}{k_{t+1}\alpha k_{t+1}^{\alpha-1} + \frac{\kappa}{\varsigma}\eta(1-\alpha)k_{t+1}^{\alpha}}\right] + (1-\varsigma)\varphi\frac{(1-s_t)}{s_t}\right) \\
&= \beta\left(\mathbb{E}_t\left[\varsigma\frac{\frac{1-s_t}{s_t}k_{t+1}\alpha k_{t+1}^{\alpha-1}(1-\tau_{t+1})}{k_{t+1}\alpha k_{t+1}^{\alpha-1} + \frac{\kappa}{\varsigma}\eta(1-\alpha)k_{t+1}^{\alpha}}\right] + (1-\varsigma)\varphi\frac{(1-s_t)}{s_t}\right) \\
&= \beta\frac{1-s_t}{s_t}\left(\mathbb{E}_t\left[\varsigma\frac{(1-\tau_{t+1})}{\alpha + (1-\alpha)\frac{\kappa}{\varsigma}\eta}\right] + (1-\varsigma)\varphi\right) \\
&= \beta\frac{1-s_t}{s_t}\left(\varsigma(1-\tau_{t+1})\Gamma(\alpha,\kappa,\varsigma,\Psi) + (1-\varsigma)\varphi\right) \\
&= \frac{1-s_t}{s_t}\Lambda(\tau_{t+1},\alpha,\beta,\kappa,\varsigma,\varphi,\Psi)
\end{aligned}$$

and thus

$$s_t = \frac{1}{1 + \Lambda(\cdot)^{-1}},$$

where $\Lambda(\tau_{t+1}, \alpha, \beta, \kappa, \varsigma, \varphi, \Psi) = \beta\left[\varsigma(1-\tau_{t+1})\Gamma(\alpha, \kappa, \varsigma, \Psi) + (1-\varsigma)\varphi\right]$ and $\Gamma(\alpha, \kappa, \varsigma, \Psi) = \int \frac{1}{\alpha + (1-\alpha)\frac{\kappa}{\varsigma}\eta} d\Psi(\eta)$. The benchmark model is obtained for $\varsigma = 1$. Therefore, the saving rate is constant $s_t = s$ for all t if and only if the capital income tax is constant $\tau_{t+1} = \tau$ for all t. Furthermore, the comparative statics results from the main paper apply to this extension unchanged, i.e., the competitive equilibrium saving rate s_t increases in income risk, and it falls in the capital tax rate τ_{t+1}. Also note that s_t increases in the bequest utility parameter φ, and since

$$\frac{\partial \Lambda}{\partial \varsigma} = \beta(1-\tau_{t+1})\left(\Gamma + \varsigma \underbrace{\frac{\partial \Gamma}{\partial \varsigma}}_{>0}\right) - \beta\varphi$$

the saving rate increases in survival risk ς only if the bequest utility parameter φ is sufficiently low. Otherwise, leaving warm-glow bequests is so valuable, in utility terms, that a higher likelihood of death *increases* savings incentives.

As noted above τ_{t+1} must be strictly less than one for the maximization problem of the household to be well-defined and a competitive equilibrium to exist. This implies a lower bound on the set of implementable saving rates which we can derive from the private household first-order condition, by solving for $1 - \tau_{t+1}$

$$\begin{aligned}
& 1 = \beta \frac{1-s_t}{s_t}\left[\varsigma(1-\tau_{t+1})\Gamma(\alpha, \kappa, \varsigma, \Psi) + (1-\varsigma)\varphi\right] \\
\Leftrightarrow \qquad & 1 - \tau_{t+1} = \frac{1}{\varsigma\Gamma}\left(\frac{s_t}{\beta(1-s_t)} - (1-\varsigma)\varphi\right) \\
& = \frac{1}{\varsigma\Gamma}\frac{s_t - \beta(1-\varsigma)\varphi(1-s_t)}{\beta(1-s_t)} \\
& = \frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi) - \beta(1-\varsigma)\varphi}{\beta(1-s_t)}
\end{aligned}$$

and by next noting that

$$
\begin{aligned}
& 1-\tau_{t+1}>0 \\
\Leftrightarrow \quad & \frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s_t)}>0 \\
\Leftrightarrow \quad & s_t>\frac{1}{1+(\beta(1-\varsigma)\varphi)^{-1}}.
\end{aligned} \tag{69}
$$

Finally, note that to implement a saving rate approaching 1 we require a tax rate

$$\lim_{s_t\to 1} 1-\tau_{t+1}=\lim_{s_t\to 1}\frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s_t)}=+\infty$$

and thus a $\tau_{t+1}=-\infty$ is required. This leads to the following proposition, the counterpart to proposition 1 of the main text:

Proposition 12. *For all $k_t>0$ and all $\tau_t\in(-\infty,1)$ the unique saving rate is given by*

$$s_t=\frac{1}{1+\Lambda(\cdot)^{-1}}\in\left(\frac{1}{1+(\beta(1-\varsigma)\varphi)^{-1}},1\right)$$

where $\Lambda(\tau_{t+1},\alpha,\beta,\kappa,\varsigma,\varphi,\Psi)=\beta\left[\varsigma(1-\tau_{t+1})\Gamma(\alpha,\kappa,\varsigma,\Psi)+(1-\varsigma)\varphi\right]$ and $\Gamma(\alpha,\kappa,\varsigma,\Psi)=\int\frac{1}{\alpha+(1-\alpha)\frac{\kappa}{\varsigma}\eta}d\Psi(\eta)$.

G.3.6 Maximizing Steady State Utility

In order to obtain a sharp characterization of the optimal solution to the Ramsey problem with warm-glow bequests we focus on the case of $\theta=1$ in which case the Ramsey government maximizes steady state welfare. To do so, we now rewrite the Ramsey problem in terms of the steady state capital stock $k(s)$, which in turn is determined by the steady state saving rate s. To this purpose note that

$$
\begin{aligned}
a_t^y+T_t^y &= (1-\varsigma)a_t^oR_t(1-\tau_t)+(1-\varsigma)a_t^oR_t\tau_t \\
&= (1-\varsigma)a_t^oR_t
\end{aligned}
$$

and thus in general equilibrium

$$a_t^y+T_t^y=(1-\varsigma)a_t^oR_t=(1-\varsigma)k_t\alpha k_t^{\alpha-1}=(1-\varsigma)\alpha k_t^{\alpha}$$

and thus consumption of the young and old in general equilibrium is

$$\begin{aligned}
c_t^y &= (1-s_t)x_t = (1-s_t)\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)k_t^\alpha \\
c_{t+1}^o(\eta) &= \left(\alpha+(1-\alpha)\frac{\kappa}{\varsigma}\eta\right)k_{t+1}^\alpha.
\end{aligned}$$

Similarly, we can write bequeathed wealth, including net-of-tax interest, as

$$\begin{aligned}
a_{t+1}^o R_{t+1}(1-\tau_{t+1}) &= k_{t+1}\alpha k_{t+1}^{\alpha-1}(1-\tau_{t+1}) \\
&= \alpha k_{t+1}^\alpha(1-\tau_{t+1}).
\end{aligned}$$

Recall from the implementation result that

$$1-\tau_{t+1} = \frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s_t)}$$

and thus bequeathed wealth in terms of the saving rate in period t and the capital stock in period $t+1$ is

$$\begin{aligned}
a_{t+1}^o R_{t+1}(1-\tau_{t+1}) &= \alpha k_{t+1}^\alpha(1-\tau_{t+1}) \\
&= \alpha\frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s_t)}k_{t+1}^\alpha.
\end{aligned}$$

From the implementation result in Proposition 12 it follows that $s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi > 0$ and thus $a_{t+1}^o R_{t+1}(1-\tau_{t+1}) > 0$.

The link between the saving rate and the capital stock is

$$\begin{aligned}
k_{t+1} &= s_t x_t \\
&= s_t\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)k_t^\alpha,
\end{aligned}$$

and thus the steady state capital stock, as a function of the steady state saving rate s, is

$$k(s) = \left[s\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)\right]^{\frac{1}{1-\alpha}}.$$

We can then rewrite consumption when young and old and bequeathed wealth in terms of the steady state capital stock and the steady state saving rate s as

$$\begin{aligned} c^y &= (1-s)\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)k(s)^\alpha \\ c^o(\eta) &= \left(\alpha+(1-\alpha)\frac{\kappa}{\varsigma}\eta\right)k(s)^\alpha \\ a^oR(1-\tau) &= \alpha\frac{1}{\varsigma\Gamma}\frac{s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s)}k(s)^\alpha \end{aligned}$$

The social welfare function maximizing steady state utility in terms of the saving rate s is therefore given by

$$\begin{aligned} W(s) &= \ln(c^y)+\beta\varsigma\int\ln(c^o(\eta))d\Psi(\eta)+\beta(1-\varsigma)\varphi\ln\left(a^oR(1-\tau)\right) \\ &= \ln\left[(1-s)\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)k(s)^\alpha\right]+ \\ &\quad \beta\varsigma\int\ln\left[\left(\alpha+(1-\alpha)\frac{\kappa}{\varsigma}\eta\right)k(s)^\alpha\right]d\Psi(\eta)+ \\ &\quad \beta(1-\varsigma)\varphi\ln\left[\alpha\frac{1}{\varsigma\Gamma}\frac{s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s)}k(s)^\alpha\right] \\ &= \Xi+\ln(1-s)+\alpha\ln(k(s))+\alpha\beta\varsigma\ln(k(s))+ \\ &\quad \alpha\beta(1-\varsigma)\varphi\ln(k(s))+\beta(1-\varsigma)\varphi\ln(s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi\ln(1-s) \\ &= \Xi+(1-\beta(1-\varsigma)\varphi)\ln(1-s)+\alpha\left(1+\beta\left(\varsigma+(1-\varsigma)\varphi\right)\right)\ln(k(s))+ \\ &\quad \beta(1-\varsigma)\varphi\ln\left(s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi\right) \\ &= \tilde{\Xi}+(1-\beta(1-\varsigma)\varphi)\ln(1-s)+\frac{\alpha\left(1+\beta\left(\varsigma+(1-\varsigma)\varphi\right)\right)}{1-\alpha}\ln(s)+ \\ &\quad \beta(1-\varsigma)\varphi\ln\left(s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi\right), \end{aligned}$$

for some constants Ξ and $\tilde{\Xi}$. From the last term of the objective function we observe that the optimal saving rate lies in the interval $s\in\left(\frac{1}{1+(\beta(1-\varsigma)\varphi)^{-1}},1\right)$. The first order condition with respect to s is given by

$$f(s)=-\left(1-\beta(1-\varsigma)\varphi\right)\frac{1}{1-s}+\frac{\alpha\left(1+\beta\left(\varsigma+(1-\varsigma)\varphi\right)\right)}{1-\alpha}\frac{1}{s}+\frac{\beta(1-\varsigma)\varphi(1+\beta(1-\varsigma)\varphi)}{s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}=0$$

Using assumption 3 note that $f(s)$ is continuous and $\frac{\partial f(s)}{\partial s}<0$, $\lim_{s\to 1}=-\infty$, as well as $\lim_{s\to\frac{1}{1+(\beta(1-\varsigma)\varphi)^{-1}}}f(s)=\infty$, and thus by the intermediate value there exists a unique solution

$$s^*\left(\alpha,\beta,\kappa,\varsigma,\varphi\right)\in\left(\frac{1}{1+\left(\beta(1-\varsigma)\varphi\right)^{-1}},1\right),$$

which is independent of income risk. Also recall from the implementation result in Proposition 12 that this optimal saving rate can be implemented by some tax rate $\tau^* \in (-\infty, 1)$, which is increasing in income risk. Thus our main results from the benchmark model without mortality risk and warm-glow bequests go through qualitatively unchanged, even though we can no longer solve for the optimal Ramsey saving rate in closed form.

Finally, we can establish additional comparative statics results with respect to the new parameter φ measuring the importance of warm-glow bequests. To do so rewrite $f(s)$ as

$$f(s)=-\left(1-\beta(1-\varsigma)\varphi\right)\frac{1}{1-s}+\frac{\alpha\left(1+\beta\left(\varsigma+(1-\varsigma)\varphi\right)\right)}{1-\alpha}\frac{1}{s}+\frac{\beta(1-\varsigma)\varphi}{s-\frac{1}{1+[\beta(1-\varsigma)\varphi]^{-1}}}=0$$

and note that since $s > \frac{1}{1+[\beta(1-\varsigma)\varphi]^{-1}}$ we have $\frac{\partial f(s)}{\partial\varphi} > 0$ and thus by the implicit function theorem

$$\frac{\partial s}{\partial\varphi}=-\frac{\frac{\partial f(s)}{\partial\varphi}}{\frac{\partial f(s)}{\partial s}}>0.$$

Therefore, s^* is increasing in the bequest utility weight parameter φ (as is the optimal competitive equilibrium saving rate s^{CE}).

G.4 One-Sided Altruism

Finally, in this subsection we discuss a model where private intergenerational transfers are motivated by one-sided altruism of parents towards their children. Thus, rather than valuing bequests directly in the utility functions parents value the lifetime utility of their children and potentially give bequests in order to raise that lifetime utility. We aim to show that this model shares strong similarities to an Aiyagari (1994) style model with infinitely lived agents facing uninsurable idiosyncratic income risk whose optimal fiscal policy implications are explored by the references cited in the introduction.

To this end, consider an economy that again extends from $t = 0, 1, \ldots$. Young and old households are intergenerationally linked through one-sided altruism whose strength is governed by the parameter $\delta \geq 0$, which measures the relative weight on the lifetime utility of the offspring in the lifetime utility function of the parental generation.

G.4.1 Budget Constraints

We assume that bequests b_t flow from the current period old to the current period young within the period so that no interest payments accrue on bequests. Since b_t denotes the private transfers received by generation t when young, accordingly b_{t+1} is the private transfer this generation pays as bequest to the currently young in $t+1$. We follow the literature (e.g., Bernheim (1989)) and assume that inter-generational transfers cannot be negative, $b_t \geq 0$. We also augment the model with a standard borrowing constraint. Since now, through intergenerational linkages, we obtain an endogenous wealth distribution (see below), households are no longer ex-ante identical when young and this borrowing constraint is potentially binding for some households. We set the borrowing constraint to $-\bar{A}_t \geq -\bar{A}_t^{NB}$, where $-\bar{A}_t^{NB} = -\frac{\kappa \underline{\eta}}{R_{t+1}(1-\tau_{t+1})}$ is the natural debt limit, with $\underline{\eta} = \min \eta$. The budget constraints for cohort t read as

$$c_t^y + a_{t+1} = (1-\kappa)w_t + b_t = x_t^y \tag{70a}$$

$$c_{t+1}^o + b_{t+1} = a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1} = x_{t+1}^o \tag{70b}$$

$$a_{t+1} \geq -\bar{A}_t \tag{70c}$$

$$b_{t+1} \geq 0, \tag{70d}$$

where the timing of action is such that consumption c_{t+1}^o and transfers b_{t+1} take place after the income shock η_{t+1} has been realized. We define by x_t^y cash-in-hand of young and by x_t^o cash-in-hand of old households and note that the law of motion for cash-in-hand is given by

$$x_{t+1}^o = \left(x_t^y - c_t^y\right) R_{t+1}(1-\tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1}. \tag{71}$$

Adding the budget constraints of the young and old households in period t we obtain

$$c_t + a_{t+1} = a_t R_t(1-\tau_t) + (1-\kappa+\kappa\eta_t)\, w_t + T_t$$

where $c_t = c_t^y + c_t^o$ is the total consumption of a dynasty in period t. Thus, the budget constraint of the period t dynastic household is equivalent to a standard budget constraint in an Aiyagari (1994) style model with idiosyncratic income risk where the income shock is $\epsilon_t = (1-\kappa) + \kappa\eta_t$.

G.4.2 Preferences

We denote preferences of cohort $t = -1, \ldots, \infty$ by $\tilde{V}_t$. Each cohort t takes as given future cohorts' optimal decision rules $c^y_{t+s}(x^y_{t+s})$, $c^o_{t+s}(x^o_{t+s})$, $a^o_{t+s+1}(x^y_{t+s})$, $b_{t+s+1}(x^o_{t+s+1})$, for $s > t$. Preferences of cohort t are given by

$$\begin{aligned}\tilde{V}_t &= u(c^y_t) + E_t\left[\beta u(c^o_{t+1}) + \delta\left(u(c^y_{t+1}(x^y_{t+1})) + \beta u(c^o_{t+2}(x^o_{t+2}))\right)\right.\\ &\qquad \left.+\delta^2\left(u(c^y_{t+2}(x^y_{t+2})) + \beta u(c^o_{t+3}(x^o_{t+3}))\right) + \ldots\right]\\ &= u(c^y_t) + E_t\left[\delta\left(u(c^y_{t+1}(x^y_{t+1})) + \frac{\beta}{\delta}u(c^o_{t+1})\right) + \right.\\ &\qquad \left.\delta^2\left(u(c^y_{t+2}(x^y_{t+2})) + \frac{\beta}{\delta}u(c^o_{t+2}(x^o_{t+2}))\right) + \ldots\right], \end{aligned} \tag{72}$$

where expectations in t are taken with respect to the sequence of shocks $\{\eta_s\}_{s=t+1}^{\infty}$. Observe that in any period $s > t$ the relative utility weight between the old and young is $\frac{\beta}{\delta}$.

We assume that the initial old cohort alive in period 0 have the same preferences but its consumption-savings decision at period -1 has already been made and thus remaining per period 0 utility constitutes a constant that cannot be affected by the policy instruments available to the Ramsey government. We spell out the maximization problem of the initial old explicitly in the next subsection.

G.4.3 The Dynastic Competitive Equilibrium

We focus on a sequential competitive equilibrium where in period 0 all dynastic households are identical. This is achieved by setting $\eta_0 = 1$ so that the initial old are (ex-post) identical. Thus, the distribution of cash-in-hand of the old $\Phi^o_0(x_0)$ is degenerate with unit mass at $x^o_0 = a_0R_0(1-\tau_0) + \kappa\eta_0 w_0 + T_0$, and, by market clearing in the capital market, we have $a_0 = k_0$. As a consequence, initial consumption c^o_0 and initial transfers of the old households to the period 0 young households $b_0 \geq 0$ are singletons, and emerge from maximizing

$$\tilde{V}_{-1} = \left[u(c^y_0(x^y_0)) + \frac{\beta}{\delta}u(c^o_0)\right] + \delta E_0\left[u(c^y_1(x^y_1)) + \frac{\beta}{\delta}u(c^o_1(x^o_1)) + \ldots\right] \tag{73}$$

subject to the constraints

$$c_0^o + b_0 = a_0 R_0(1-\tau_0) + \kappa w_0 + T_0 = x_0^o \quad \text{and} \quad b_0 \geq 0 \tag{74}$$

taking as given future cohorts' optimal decision rules $c_s^y(x_s^y)$, $c_s^o(x_s^o)$, $a_{s+1}^o(x_s^y)$, $b_{s+1}(x_{s+1}^o)$, for all $s > 0$. Since there is no transfer heterogeneity among the initial old, the period 0 young are ex-ante identical and the endogenously determined distribution of the initial young $\Phi_0^y(x_0^y)$ is degenerate, with a unit mass of cash-in hand equal to $x_0^y = (1-\kappa)w_0 + b_0$. Equipped with these initial conditions we set the stage for the social welfare function defined below, which, as in Davila et al. (2012) and the optimal Ramsey policy literature in Bewley-style models, evaluates welfare in a sequential equilibrium from an ex-ante perspective where all households are identical.

For a given policy, a sequential dynastic competitive equilibrium is defined as follows:

Definition 2. *Given the initial condition $k_0 = a_0$, and an associated degenerate initial distribution $\Phi^o(x_0)$ with unit mass at $x_0 = a_0 R_0(1-\tau_0) + \kappa w_0 + T_0$ and a sequence of tax policies $\{\tau_t\}_{t=0}^{\infty}$ a dynastic competitive equilibrium is an allocation $\{c_t^y, c_t^o, L_t, a_{t+1}, b_{t+1}, x_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, cross-sectional measures $\{\{\Phi_t^j(x_t)\}_{j\in\{y,o\}}\}_{t=0}^{\infty}$, prices $\{R_t, w_t\}_{t=0}^{\infty}$ and transfers $\{T_t\}_{t=0}^{\infty}$ such that*

1. *given prices $\{R_t, w_t\}_{t=0}^{\infty}$ and government policies $\{\tau_t, T_t\}_{t=0}^{\infty}$, for each $t \geq 0$,*
   - *(a) consumption, savings and transfer decisions $(c_t^y(x_t^y), c_{t+1}^o(x_{t+1}^o), a_{t+1}(x_t^y), b_{t+1}(x_{t+1}^o))$ maximize (72) subject to (70), and households take as given optimal decision rules at $s > t$, $(c_{t+s}^y(x_{t+s}^y), c_{t+s}^o(x_{t+s}^o), a_{t+s+1}^o(x_{t+s}^y), b_{t+s+1}(x_{t+s+1}^o))$;*
   - *(b) consumption and transfers of the initial old ex-post identical households $(c_0^o(x_0^o), b_0(x_0^o))$ follow from maximizing (73) subject to (74) taking as given future cohorts' optimal decision rules at $s > 0$, $c_s^y(x_s^y), c_s^o(x_s^o), a_{s+1}^o(x_s^y), b_{s+1}(x_{s+1}^o)$;*
2. *prices satisfy equations (3a) and (3b);*
3. *the government budget constraint is satisfied in every period: for all $t \geq 0$*

$$T_t = \tau_t R_t k_t$$

4. *markets clear*

$$\begin{aligned} L_t &= L = 1 \\ k_{t+1} &= \int a_{t+1}(x_t^y) d\Phi_t^y(x_t^y) \\ C_t^j &= \int c_t^j(x_t^j) d\Phi_t^j(x_t^j), \textit{ for } j \in \{y, o\} \\ C_t^y + C_t^o + k_{t+1} &= k_t^\alpha \end{aligned}$$

5. *the cross sectional measures evolve as*

$$\begin{aligned} \Phi_{t+1}^o(x_{t+1}^o) = H^o(\Phi_t^y(x_t^y)) \\ \Phi_t^y(x_t^y) = H^y(\Phi_t^o(x_t^o)), \end{aligned}$$

where the law of motion H^o is generated by the cash-in-hand transition (71) and the stochastic i.i.d. shocks $\eta_{t+1} \sim \Psi(\eta_{t+1})$, and the law of motion H^y is generated by the transfer decision $b_t(x_t^o)$.

In a dynastic competitive equilibrium the consumption-savings-transfer problem of any cohort t is solved by backward induction, starting from a final steady state. Denote by $a_{t+2}(x_{t+1}^y(b_{t+1}))$ the savings decision function of cohort $t+1$, for a given amount of transfers b_{t+1}, which we make explicit by writing $x_{t+1}^y(b_{t+1})$. Use (70) in (72) to get

$$\tilde{V}_t = u(x_t^y - a_{t+1}) + \delta E_t \left[u\left(\underbrace{(1-\kappa)w_{t+1} + b_{t+1} - a_{t+2}(x_{t+1}^y(b_{t+1}))}_{=c_{t+1}^y(x_{t+1}^y)} \right) + \frac{\beta}{\delta} u\left(\underbrace{x_{t+1}^o - b_{t+1}}_{=c_{t+1}^o(x_{t+1}^o)} \right) + \dots \right],$$

which shows how a period t cohort influences, through its transfer decision b_{t+1}, the consumption-savings decision of its successor's generation when young $c_{t+1}^y(x_{t+1}(b_{t+1}))$. As in a standard consumption savings model with a borrowing constraint, constraints (70c) and (70d) induce a precautionary savings motive beyond the standard prudence motive, because a binding constraint (70c) in period $t+1$ will reduce c_{t+1}^y, and a binding constraint (70d) will reduce c_{t+1}^o, relative to the optimal interior paths. These occasionally binding constraints, together with the standard prudence argument, will induce households to save more in period t in the presence of idiosyncratic income risk, in turn inducing the pecuniary externality from changing factor prices w_{t+1}, R_{t+1} in general equilibrium emphasized in the main text.

G.4.4 Social Welfare Function

As discussed above, we follow Davila et al. (2012) and evaluate welfare from an ex-ante perspective. Consider a Ramsey government that weighs explicitly the utility of the period zero young and all future generations through Pareto weights $\rho_t \geq 0$ and that—different from Bernheim (1989)—, also puts welfare weight $\rho_{-1} \geq 0$ on the initial old generation's per period utility:

$$SWF = \rho_{-1} u(c_0^o) + \rho_0 \tilde{V}_0 + E_0 \left[\sum_{t=1}^{\infty} \rho_t \tilde{V}_t \right]$$

where we note that each term in the above infinite sum takes the form

$$\begin{aligned} E_0[\tilde{V}_t] &= E_0 \left[u(c_t^y) + E_t \left[\delta \left(u(c_{t+1}^y(x_{t+1}^y)) + \frac{\beta}{\delta} u(c_{t+1}^o) \right) + \delta^2 \left(u(c_{t+2}^y(x_{t+2}^y)) + \frac{\beta}{\delta} u(c_{t+2}^o(x_{t+2}^o)) \right) + \dots \right] \right] \\ &= E_0 \left[u(c_t^y) + \delta \left(u(c_{t+1}^y(x_{t+1}^y)) + \frac{\beta}{\delta} u(c_{t+1}^o) \right) + \delta^2 \left(u(c_{t+2}^y(x_{t+2}^y)) + \frac{\beta}{\delta} u(c_{t+2}^o(x_{t+2}^o)) \right) + \dots \right] \end{aligned}$$

by the law of iterated expectations. We can thus rewrite the social welfare function as

$$\begin{aligned} SWF &= \rho_{-1} u(c_0^o) + E_0 \left[\rho_0 \left(u(c_0^y) + \beta u(c_1^o) + \delta \left(u(c_1^y) + \beta u(c_2^o) \right) + \delta^2 \left(u(c_2^y) + \beta u(c_3^o) \right) + \dots \right) + \right. \\ &\qquad \left. + \rho_1 \left(u(c_1^y) + \beta u(c_2^o) + \delta \left(u(c_2^y) + \beta u(c_3^o) \right) + \dots \right) + \dots \right] \\ &= \rho_{-1} u(c_0^o) + \rho_0 V_0 + E_0 \left[\left(\rho_0 \delta + \rho_1 \right) V_1 + \left(\rho_0 \delta^2 + \rho_1 \delta + \rho_2 \right) V_2 + \dots \right] \\ &= \rho_{-1} u(c_0^o) + \omega_0 V_0 + E_0 \left[\sum_{t=1}^{\infty} \omega_t V_t \right] \end{aligned} \tag{75}$$

V_t is expected lifetime utility of generation t, $\omega_t = \sum_{s=0}^{t} \rho_s \delta^{t-s}$ and $\sum_{t=0}^{\infty} \omega_t < \infty$ is assumed.

First assume that inter-generational transfers are not operational so that $b_t = 0$ in all t. Then all households in all periods t start with zero bequests, are ex-ante identical and the borrowing constraint (70c) is not binding. Since the social welfare function (75) is the same as the one in (4) all results in the main text on the Ramsey optimum can therefore, not surprisingly, be reinterpreted as emerging in a dynastic competitive market economy where intergenerational transfers are not operative.

Now return to the general case where intergenerational transfers are potentially operative so that $b_t \geq 0$ for all $t \geq 0$. Assume first that $\rho_t = 1$ for $t = 0$ and $\rho_t = 0$ for all $t > 0$. Then the Ramsey government maximizes the same objective as the dynastic period 0 household in competitive equilibrium. In this case $\omega_t = \delta^t$ and the social welfare function is recursive, as in the benchmark model of the main text. The Ramsey government internalizes two effects not taken into account by dynastic households making private consumption-saving decisions in the competitive equilibrium. First as in the benchmark model there is a pecuniary externality from increasing savings on the equilibrium wage and interest rate. The increase in the wage raises the stochastic income component in old age, the lower interest rate may lead to increased borrowing (if the substitution and the human capital wealth effect dominate the income effect) and thus more frequently binding borrowing constraints (70c) and (70d). Second, households do not internalize the distributional effects their decisions have, through changing factor prices, on the endogenously evolving wealth distribution. These are precisely the same mechanisms emphasized in Davila et al. (2012)'s study of the constrained planner problem of the Aiyagari (1994) model.

If, in addition $\rho_t > 0$ for $t > 0$ then the Ramsey government puts additional weight on future generations and thus, in addition to these two mechanisms, the future generations effect from the OLG benchmark model of the main text is operative. In this case we can engineer welfare weights ρ_t such that the social welfare function again has a recursive representation, but now with $\omega_t = \theta^t$. Concretely, this construction is given as follows: using that $\omega_t = \sum_{s=0}^{t} \rho_s \delta^{t-s}$ we obtain:

$$\begin{aligned}
&\omega_0 = \theta^0 = 1 = \rho_0 && \Leftrightarrow && \rho_0 = 1 \\
&\omega_1 = \theta = \rho_0\delta + \rho_1 && \Leftrightarrow && \rho_1 = \theta - \delta \\
&\omega_2 = \theta^2 = \delta^2 + \rho_1\delta + \rho_2 && \Leftrightarrow && \rho_2 = \theta^2 - \delta^2 - \delta(\theta - \delta) = \theta(\theta - \delta) \\
&\omega_3 = \theta^3 = \delta^3 + \rho_1\delta^2 + \rho_2\delta + \rho_3 && \Leftrightarrow && \rho_3 = \theta^3 - \delta^3 - \delta^2(\theta - \delta) - \delta\left(\theta - \delta\theta\right) = \theta^2\left(\theta - \delta\right)
\end{aligned}$$

and thus $\rho_t = 1$, for $t = 0$ and $\rho_t = \theta^{t-1}\left(\delta - \theta\right)$ for $t > 0$. Notice that $\rho_t \geq 0$ for $t > 0$ if and only if $\theta \geq \delta$. Thus, this is a valid social discount function if and only if the planner exhibits weakly more patience than the dynastic household, and we summarize the cases as:

$$\omega_t = \begin{cases} \delta^t & \text{for } \theta = \delta \\ \theta^t & \text{for } \theta > \delta, \end{cases}$$

In the first case the Ramsey government—in addition to valuing the initial old generation through weight ρ_{-1}—shares the same objective as the initial dynastic cohort at $t = 0$. In the second case, the Ramsey government values future generations more heavily and the future generations effect is operative.

To summarize the discussion in this section, we note that in the dynastic model described here the same mechanisms shape optimal Ramsey allocations as they do in the constrained planner problem of Davila et al. (2012) if the Ramsey government weighs only the period 0 utility of the period 0 dynastic household from an ex-ante perspective. If the Ramsey government places additional welfare weights on future generations, then the additional future generations effect from the main text emerges, pushing down the optimal capital income taxes. An analytical solution of this model is infeasible even with log-utility, and a full numerical exploration is left for future research, and, in the case of $\theta = \delta$, such an analysis directly relates to the papers on optimal policy in the Aiyagari (1994)-style models cited in the main text.

H Capital Stock Dynamics and Capital *Income* Taxes

In this appendix we make precise the relation between the capital taxes τ_t studied thus far, and the implied optimal capital *income* taxes τ_t^k. These are related by the equation

$$1 + (R_t - 1)(1 - \tau_t^k) = R_t(1 - \tau_t)$$

and thus

$$\tau_t^k = \frac{R_t}{R_t - 1}\tau_t,$$

where the gross return is given by $R_t = \alpha\,(k_t)^{\alpha-1}$. As long as $R_t > 1$ for all t, capital taxes and capital income taxes have the same sign. To give a sufficient condition for this, note that the saving rate, together with the law of motion for the capital stock

$$k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^\alpha = \frac{\alpha(\theta+\beta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}k_t^\alpha$$

and the initial condition k_0 determine the entire time path for the capital stock. That sequence $\{k_t\}_{t=1}^\infty$ is independent of the amount of income risk and converges monotonically to the steady state

$$k^* = \left[\frac{\alpha(\theta+\beta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}\right]^{\frac{1}{1-\alpha}},$$

either from above if $k_0 > k^*$ or from below, if $k_0 < k^*$. A sufficient condition for $R_t > 1$ for all t can then be given as:

Assumption 4. *The initial capital stock and the model parameters satisfy* $k_0 < \alpha^{\frac{1}{1-\alpha}}$ *and*

$$\frac{1+\alpha\beta}{(\theta+\beta)(1-\kappa)(1-\alpha)} > 1.$$

This assumption assures that net returns are strictly positive at all times in the Ramsey equilibrium, since $R_0 = \alpha\,(k_0)^{\alpha-1} > 1$ and $R^* = \alpha\,(k^*)^{\alpha-1} > 1$, (and because the sequence of R_t along the transition is monotone) and thus the Ramsey allocation can be supported by capital income taxes of the same sign as the corresponding wealth taxes. Under assumption 4 therefore all interpretations and qualitative results extend without change to capital income taxes.

I Robustness to Other Assumptions

I.1 Idiosyncratic Return Risk

We denote return shocks by ϱ_{t+1} and assume that they are iid. We assume that the cdf of ϱ is given by $\Upsilon(\varrho)$ and denote the corresponding pdf by $\upsilon\,(\varrho)$. We again assume that a LLN applies so that Υ is both the population distribution of ϱ as well as the individual cdf of return shocks. We make the following

Assumption 5. *The shock* ϱ *takes positive values* Υ*-almost surely and*

$$\int \varrho d\Upsilon = 1.$$

Furthermore, shocks η *and* ϱ *are independent*[12] *and therefore*

$$\int_{\varrho}\int_{\eta} \varrho\eta d\Upsilon(\varrho)d\Psi(\eta) = \int \varrho d\Upsilon(\varrho)\cdot\int \eta d\Psi(\eta)$$

almost surely.

[12]Independence is assumed for simplicity of notation but can be relaxed for the result.

The budget constraints now write as

$$\begin{aligned} a_{t+1} + c_t^y &= (1-\kappa)w_t \\ c_{t+1}^o(\eta,\varrho) &= a_{t+1}R_{t+1}\varrho_{t+1}(1-\tau_{t+1}) + \eta_{t+1}\kappa w_{t+1} + T_{t+1}(\varrho) \end{aligned}$$

and we assume that transfer payments are contingent on the rate of return realization,

$$T_{t+1}(\varrho) = a_{t+1}R_{t+1}\varrho_{t+1}\tau_{t+1}.$$

I.1.1 General Equilibrium

Proposition 13. *The structure of the competitive equilibrium is unchanged, but now idiosyncratic risk summarized by* Γ *is expressed in terms of the distribution* $\Pi(\delta_{t+1})$ *of the random variable* $\delta_{t+1} = \frac{\eta_{t+1}}{\varrho_{t+1}}$ *instead of* $\Psi(\eta_{t+1})$.

Proof. The first-order condition for log utility is now

$$\begin{aligned} 1 &= \beta R_{t+1}(1-\tau_{t+1}) \int\int \varrho_{t+1}\frac{c_t^y}{c_{t+1}^o(\eta)} d\Psi(\eta)d\Upsilon(\varrho) \\ &= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1}) \int\int \varrho_{t+1}\frac{(1-s_t)(1-\kappa)(1-\alpha)k_t^\alpha}{s_t(1-\kappa)(1-\alpha)k_t^\alpha \alpha k_{t+1}^{\alpha-1}\varrho_{t+1} + \eta_{t+1}\kappa(1-\alpha)k_{t+1}^\alpha} d\Psi(\eta)d\Upsilon(\varrho) \\ &= \alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\int\int \left(\alpha + \kappa(1-\alpha)\frac{\eta_{t+1}}{\varrho_{t+1}}\right)^{-1} d\Psi(\eta)d\Upsilon(\varrho) \\ &= \alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\int \left(\alpha + \kappa(1-\alpha)\delta_{t+1}\right)^{-1} d\Pi(\delta) \\ &= \alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\Gamma(\alpha,\kappa;\delta,\Pi). \end{aligned}$$

and thus the general equilibrium saving rate is the same as before, with Γ expressed in terms of random variable δ and its cdf $\Pi(\delta)$. □

I.1.2 Ramsey Problem

Proposition 14. *The structure of the optimal Ramsey problem is unchanged, but with idiosyncratic risk now expressed in terms of the random variable* $\delta_{t+1} = \frac{\eta_{t+1}}{\varrho_{t+1}}$ *instead of* η_{t+1}.

Proof. The steps are identical to the ones in the derivation in equation (2). The objective function of the Ramsey government in the current period can be written as

$$\begin{aligned}
W(k) &= \ln\left((1-s)(1-\kappa)(1-\alpha)k^{\alpha}\right) + \beta \int\int \ln\left(\kappa\eta w(s) + R(s)\varrho k'(s)\right) d\Upsilon(\varrho) d\Psi(\eta) \\
&= \ln\left((1-s)(1-\kappa)(1-\alpha)k^{\alpha}\right) + \beta \int\int \ln\left(\varrho\left(\kappa\frac{\eta}{\varrho}(1-\alpha)+\alpha\right)k'(s)\right) d\Upsilon(\varrho) d\Psi(\eta) \\
&= \ln(1-s) + \alpha\beta\ln(s) + (1+\alpha\beta)\ln((1-\kappa)(1-\alpha)) + \alpha(1+\alpha\beta)\ln(k) \\
&+ \beta\int \ln(\varrho) d\Upsilon(\varrho) + \beta\int \ln\left(\kappa\delta(1-\alpha)+\alpha\right) d\Pi(\delta).
\end{aligned}$$

Note that the risk terms in the last line simply add maximization-irrelevant constants to the period objective of the Ramsey government. □

I.2 Ex-Ante Heterogeneity

Permanent productivity is denoted by ν and we assume that the cdf of ν is given by $\Phi(\nu)$. We assume that a LLN applies so that Φ is both the population distribution of permanent productivity ν as well as the ex-ante cdf over ν for each household. We make the following

Assumption 6. *The shock ν takes positive values Φ-almost surely and*

$$\int \nu d\Phi = 1.$$

Furthermore, shocks η and ν are independent, thus

$$\int_{\nu}\int_{\eta} \nu\eta d\Phi(\nu) d\Psi(\eta) = \int \nu d\Phi(\nu) \cdot \int \eta d\Psi(\eta) = 1.$$

The budget constraints of each household of productivity type i is now given by

$$\begin{aligned}
a_{t+1}(\nu) + c_t^y(\nu) &= (1-\kappa)\nu w_t \\
c_{t+1}^o(\nu,\eta) &= a_{t+1}R_{t+1}(1-\tau_{t+1}) + \eta_{t+1}\nu\kappa w_{t+1} + T_{t+1}(\nu),
\end{aligned}$$

where

$$T_{t+1}(\nu) = a_{t+1}(\nu)R_{t+1}\tau_{t+1}$$

In all periods t we have $L_t = \int\int\left((1-\kappa)\nu + \kappa\nu\eta_t\right) d\Psi(\eta) d\Phi(\nu) = 1$ and thus the capital stock in period $t+1$, K_{t+1}, is equal to the capital intensity $k_{t+1} = \frac{K_{t+1}}{L_{t+1}}$. Denote by

$$s_t(\nu) = \frac{a_{t+1}(\nu)}{(1-\kappa)\nu w_t}$$

the saving rate of household of type ν. The capital intensity in period $t+1$ is then

$$k_{t+1} = \int a_{t+1}(\nu) d\Phi(\nu) = (1-\kappa)(1-\alpha)k_t^\alpha \int s_t(\nu)\nu d\Phi(\nu).$$

I.2.1 General Equilibrium

Proposition 15. *The general equilibrium saving rates $s_t(\nu)$ are identical for all agents: $s_t(\nu) = s_t$ for all ν.*

Proof. If $s(\nu) = s_t$ then since $\int \nu d\Phi(\nu) = 1$ the law of motion of the capital stock is

$$k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^\alpha.$$

The first-order condition with log utility of each household is now

$$\begin{aligned}
1 &= \beta R_{t+1}(1-\tau_{t+1}) \int\int \frac{c_t^y(\nu)}{c_{t+1}^o(\eta,\nu)} d\Psi(\eta) d\Phi(\nu) \\
&= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1}) \int\int \frac{(1-s_t)\nu(1-\kappa)(1-\alpha)k_t^\alpha}{s_t(1-\kappa)(1-\alpha)k_t^\alpha \alpha k_{t+1}^{\alpha-1} + \eta_{t+1}\kappa(1-\alpha)k_{t+1}^\alpha} d\Psi(\eta) d\Phi(\nu) \\
&= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1}) \int \frac{\frac{(1-s_t)}{s_t} k_{t+1}}{k_{t+1}\alpha k_{t+1}^{\alpha-1} + \eta_{t+1}\kappa(1-\alpha)k_{t+1}^\alpha} d\Psi(\eta) \\
&= \alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\Gamma.
\end{aligned}$$

Thus the optimal saving rate is independent of permanent productivity ν. □

I.2.2 Ramsey Problem

Proposition 16. *Permanent ex-ante heterogeneity in productivity ν does not affect the optimal choice of s.*

Proof. The objective of the Ramsey planner is now given by

$$
\begin{aligned}
W(k) = \max_{s\in(0,1)} \int \ln\left((1-s)\nu(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Phi(\nu) + \\
\beta \int\int \ln\left(\kappa\eta\nu w(s) + R(s)s\nu(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Phi(\nu) d\Psi(\eta), \\
= (1+\beta)\int \ln(\nu) d\Phi(\nu) + \max_{s\in(0,1)} \ln\left((1-s)(1-\kappa)(1-\alpha)k^{\alpha}\right) + \\
\beta \int \ln\left(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Psi(\eta)
\end{aligned}
$$

and thus heterogeneity with respect to ν does not affect the optimization. □

I.3 Time Varying Technological Progress and Population Growth

Denote by A_t the level of technology (labor productivity) and assume that it evolves deterministically according to $A_t = (1+g_t)A_{t-1}$, where the growth rate of technology g_t is allowed to be time-varying. The population growth rate $n \geq 0$ is assumed to be constant over time, so that the size of the young population evolves according to $N_t^y = (1+n)N_{t-1}^y$. With these modifications, aggregate production is

$$
Y_t = F(K_t, A_t L_t) = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha},
$$

where L_t is aggregate labor supply given by

$$
L_t = (1-\kappa)N_t^y + \kappa N_t^o = \left((1-\kappa)(1+n) + \kappa\right) N_{t-1}^y.
$$

Define the capital intensity in terms of efficiency units of labor as $k_t = \frac{K_t}{A_t L_t}$. Then, under the maintained assumption of Cobb-Douglas production, $Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$ we get $y_t = \frac{Y_t}{A_t L_t} = k_t^{\alpha}$ and thus wages (per effective unit of labor) and interest rates are

$$
\begin{aligned}
w_t &= (1-\alpha)k_t^{\alpha} A_t \\
R_t &= \alpha k_t^{\alpha-1}.
\end{aligned}
$$

The law of motion of the capital intensity can be derived as

$$\begin{aligned} K_{t+1} &= a_{t+1} N_t^y = s_t(1-\kappa)(1-\alpha) k_t^\alpha A_t N_t^y \\ \Leftrightarrow \qquad k_{t+1} &= s_t \frac{(1-\kappa)(1-\alpha)}{(1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)} k_t^\alpha . \end{aligned}$$

I.3.1 General Equilibrium

Proposition 17. *A time varying rate of technological progress g_t does not affect the saving rate in the competitive general equilibrium, whereas an increase of the constant population growth rate n increases the saving rate.*

Proof. Start from the FOC, equation (6), given by

$$1 = \beta(1-\tau_{t+1}) \int \frac{1-s_t}{s_t(1-\tau_{t+1}) + \frac{\kappa w_{t+1}}{(1-\kappa) w_t R_{t+1}} \eta_{t+1} + \frac{T_{t+1}}{(1-\kappa) w_t R_{t+1}}} d\Psi(\eta_{t+1})$$

and use that

$$\tau_{t+1} s_t = \frac{T_{t+1}}{(1-\kappa) w_t R_{t+1}}$$

to obtain

$$1 = \beta(1-\tau_{t+1}) \int \frac{1-s_t}{s_t + \frac{\kappa w_{t+1}}{(1-\kappa) w_t R_{t+1}} \eta_{t+1}} d\Psi(\eta_{t+1})$$

Next, rewrite $\frac{w_{t+1}}{w_t R_{t+1}}$ as

$$\begin{aligned} \frac{w_{t+1}}{w_t R_{t+1}} &= \frac{k_{t+1}^\alpha A_{t+1}}{k_t^\alpha A_t \alpha k_{t+1}^{\alpha-1}} = (1+g_{t+1}) \frac{1}{\alpha} \frac{k_{t+1}}{k_t^\alpha} \\ &= (1+g_{t+1}) \frac{1}{\alpha} s_t (1-\kappa)(1-\alpha) \frac{1}{(1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)} \\ &= \frac{1}{\alpha} s_t (1-\kappa)(1-\alpha) \frac{1}{(1-\kappa)(1+n)+\kappa}. \end{aligned}$$

Observe that the time varying growth rate g_{t+1} cancels out, and we can rewrite the FOC as

$$
\begin{aligned}
1 &= \alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\int \frac{1}{\alpha+\kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}\eta_{t+1}}d\Psi(\eta_{t+1}) \\
&= \alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\check{\Gamma}.
\end{aligned}
$$

where $\check{\Gamma} := \int \frac{1}{\alpha+\kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}\eta_{t+1}}d\Psi(\eta_{t+1})$. □

I.3.2 Ramsey Optimum

Proposition 18. *A time varying rate of technological progress g_t as well as a constant population growth rate n leave the optimal Ramsey saving rate unchanged.*

Proof. With log utility, cohort t lifetime utility is given by

$$
\begin{aligned}
V_t(k_t,s_t,A_t) &= \ln(A_t) + \ln\left((1-s_t)(1-\kappa)k_t^{\alpha}\right) + \alpha\beta\ln\left((1+g_{t+1})k_{t+1}(s_t)\right) + \beta\ln\left(\Gamma_2\right) \\
&= \ln(A_t) + \alpha\beta\ln(1+g_{t+1}) + \tilde{V}_t(k_t,s_t),
\end{aligned}
$$

where $\Gamma_2 = \int \left((1-\alpha)\kappa\eta_{t+1}+\alpha\right)^{1-\sigma} d\Psi(\eta_{t+1})$. Next, assume that the government maximizes the discounted sum of utility of cohorts t weighted by the population size of that cohort so that the objective is to maximize

$$
W_0 = \sum_{t=0}^{\infty}\omega_t N_t^y V_t(k_t,s_t,A_t) = \chi + \sum_{t=0}^{\infty}\omega_t N_t^y \tilde{V}_t(k_t,s_t),
$$

where χ is a maximization irrelevant constant. Finally, normalizing $N_0 = 1$ we get

$$
W_0 = \sum_{t=0}^{\infty}\tilde{\omega}_t\tilde{V}_t(k_t,s_t)
$$

where $\tilde{\omega}_t = \omega_t(1+n)^t$. Also note that

$$
k_{t+1}(s_t) = s_t\frac{(1-\kappa)(1-\alpha)}{(1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)}k_t^{\alpha}.
$$

and thus

$$
\begin{aligned}
\tilde{V}_t(k_t, s_t) &= \ln\left(((1-s_t)(1-\kappa)k_t^\alpha)\right) + \alpha\beta\ln\left(k_{t+1}(s_t)\right) + \beta\ln\left(\Gamma_2\right) \\
&= \ln\left(((1-s_t)(1-\kappa)k_t^\alpha)\right) + \alpha\beta\ln\left(s_t\frac{(1-\alpha)(1-\kappa)}{(1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)}k_t^\alpha\right) + \beta\ln\left(\Gamma_2\right) \\
&= \chi_t + \ln\left(1-s_t\right) + \alpha(1+\alpha\beta)\ln\left(k_t\right) + \alpha\beta\ln\left(s_t\right)
\end{aligned}
$$

and thus time varying technological progress and population growth only add a maximization irrelevant (time varying) additive parameter. Also since

$$
\begin{aligned}
\ln(k_{t+1}) &= \ln(1-\alpha) + \ln(1-\kappa) + \alpha\ln(k_t) + \ln(s_t) - \ln\left((1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)\right) \\
&= \varkappa_{t+1} + \sum_{\tau=0}^{t}\alpha^\tau\ln(s_{t-\tau}) + \alpha^{t+1}\ln(k_0) \\
&= \tilde{\varkappa}_{t+1} + \sum_{\tau=0}^{t}\alpha^\tau\ln(s_{t-\tau})
\end{aligned}
$$

we can substitute out $\ln\left(k_t\right)$ in the cohort t utility function (as before), which adds additional maximization irrelevant time varying terms. □

I.3.3 The Bounds of Proposition 4 with Technological Progress and Population Growth

We focus on a steady state where the rate of technological progress is a constant g.

Golden Rule. Maximizing steady state utility is equivalent to maximizing per capita consumption. The per capita resource constraint, noticing that in the social planner's optimum $c_t^o(\eta) = c_t^o$, is

$$
\frac{c_t^y N_t^y + c_t^o N_t^o}{N_t} = \frac{F(K_t, L_t) - K_{t+1}}{N_t}.
$$

Now observe that in steady state where $k_{t+1} = k_t = k$ we have

$$\begin{aligned} N_t^y &= (1+n)N_{t-1}^y, \;\; N_t^o = N_{t-1}^y \\ N_t &= N_t^y + N_t^o = (2+n)N_{t-1}^y \\ L_t &= ((1-\kappa)(1+n)+\kappa)\, N_{t-1}^y \\ F(K_t, L_t) &= k^\alpha A_t L_t = k^\alpha A_t \left((1-\kappa)(1+n)+\kappa\right) N_{t-1}^y \\ K_{t+1} &= k A_{t+1} L_{t+1} = k(1+n)(1+g)\left((1-\kappa)(1+n)+\kappa\right) N_{t-1}^y \end{aligned}$$

and thus maximizing per capita consumption is equivalent to

$$\max_k \{\tilde{c}_t^y(1+n) + \tilde{c}_t^o\} = \max_k \{(k^\alpha - k(1+n)(1+g))\left((1-\kappa)(1+n)+\kappa\right)\}$$

where $\tilde{c}_t = \frac{c_t}{A_t}$ is detrended consumption. The first-order condition gives

$$\alpha k^{\alpha-1} = (1+n)(1+g)$$

and thus the golden-rule capital stock is

$$k^{GR} = \left(\frac{\alpha}{(1+n)(1+g)}\right)^{\frac{1}{1-\alpha}}$$

with the standard intuitive explanation that, with population growth and technological progress, more efficient workers have to be equipped each period with an increasing capital stock to hold constant capital per efficient worker. The golden rule interest rate is thus

$$R^{GR} = \alpha k^{GR^{\alpha-1}} = (1+n)(1+g).$$

Finally, from the law of motion of the capital stock we have

$$k' = s\frac{(1-\kappa)(1-\alpha)}{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}k^\alpha$$

and thus the steady state capital stock for given saving rate is

$$k^* = \left(s^*\frac{(1-\kappa)(1-\alpha)}{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}\right)^{\frac{1}{1-\alpha}}.$$

Setting $k^* = k^{GR}$ then gives the golden rule saving rate as

$$s^{GR} = \frac{\alpha\left((1-\kappa)(1+n)+\kappa\right)}{(1-\kappa)(1-\alpha)(1+n)}.$$

Competitive Equilibrium and Overaccumulation of Capital Since $R^* = \alpha k^{*\alpha-1}$, the steady state interest rate for given saving rate is

$$R^* = \frac{\alpha(1+g)\left((1-\kappa)(1+n)+\kappa\right)}{s^*(1-\kappa)(1-\alpha)}.$$

Now use that

$$s^* = \frac{(1-\tau)\alpha\beta\check{\Gamma}}{1+(1-\tau)\alpha\beta\check{\Gamma}},$$

as defined above, to get

$$R^*(\tau,\check{\Gamma}) = \frac{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}{(1-\kappa)(1-\alpha)}\left(\alpha + \frac{1}{(1-\tau)\beta\check{\Gamma}}\right)$$

and thus in the laissez-faire steady state we have

$$R^*(\tau=0,\check{\Gamma}) = \frac{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}{(1-\kappa)(1-\alpha)}\left(\alpha + \frac{1}{\beta\check{\Gamma}}\right).$$

Since the laissez-faire equilibrium economy has overaccumulated capital if $R^*(\tau=0,\check{\Gamma}) < (1+n)(1+g)$ we obtain overaccumulation if

$$\begin{aligned} & \frac{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}{(1-\kappa)(1-\alpha)}\left(\alpha + \frac{1}{\beta\check{\Gamma}}\right) < (1+n)(1+g) \\ \Leftrightarrow \qquad & \beta > \frac{1}{\left(\frac{(1-\kappa)(1-\alpha)(1+n)}{(1-\kappa)(1+n)+\kappa} - \alpha\right)\check{\Gamma}} \end{aligned}$$

Recall that

$$\check{\Gamma} = \int \frac{1}{\alpha + \kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}\eta_{t+1}} d\Psi(\eta_{t+1}).$$

and thus in the deterministic economy we have

$$\bar{\bar{\check{\Gamma}}} = \frac{1}{\alpha + \kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}}$$

Now rewrite the bound on β above to get

$$\begin{aligned}
\beta &> \frac{1}{\left(\frac{(1-\kappa)(1-\alpha)(1+n)}{(1-\kappa)(1+n)+\kappa} - \alpha\right)\check{\Gamma}} \\
&= \frac{1}{\left(\frac{-\kappa(1-\alpha)+(1-\alpha)(1+n)-\kappa(1-\alpha)n}{(1-\kappa)(1+n)+\kappa} - \alpha\right)\check{\Gamma}} \\
&= \frac{1}{\left(\frac{(1-\alpha)(1+n(1-\kappa))}{1+n(1-\kappa)} - \left(\alpha + \frac{\kappa(1-\alpha)}{(1-\kappa)(1+n)+\kappa}\right)\right)\check{\Gamma}} \\
&= \frac{1}{\left((1-\alpha)\check{\Gamma} - \check{\Gamma}/\bar{\bar{\check{\Gamma}}}\right)} := \Theta_1\left(\check{\Gamma}, \bar{\bar{\check{\Gamma}}}\right)
\end{aligned}$$

Since the structure of the Ramsey problem has not changed, we continue to find that the optimal saving rate for $\theta = 1$ is

$$s^* = \frac{\alpha(1+\beta)}{1+\alpha\beta}$$

and thus the tax rate implementing it satisfies

$$1 - \tau = \frac{1+\beta}{(1-\alpha)\beta\check{\Gamma}}$$

and thus we have $\tau > 0$, if and only if

$$\frac{1+\beta}{(1-\alpha)\beta\check{\Gamma}} < 1$$

or if and only if

$$\Theta_2\left(\check{\Gamma}\right) := \frac{1}{(1-\alpha)\check{\Gamma} - 1} < \beta.$$

Stating the inequalities in terms of $\check{\Gamma}$ the regions corresponding to Proposition 4 become

1. $\check{\Gamma} > \frac{1}{\left((1-\alpha)-1/\bar{\bar{\check{\Gamma}}}\right)\beta}$: $k > k^{GR}, \tau > 0$

2. $\check{\Gamma} \in \left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1}{\left((1-\alpha)-1/\bar{\bar{\Gamma}}\right)\beta}\right]$: $k \leq k^{GR}, \tau > 0$

3. $\check{\Gamma} \in \left(\bar{\bar{\Gamma}}, \frac{1+\beta}{(1-\alpha)\beta}\right]$: $k \leq k^{GR} \tau < 0$

Recall that

$$\bar{\bar{\Gamma}} = \frac{1}{\alpha + \kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}}$$

and thus an increase of n increases $\bar{\bar{\Gamma}}$ increasing the lower bound of the third interval. By increasing $\bar{\bar{\Gamma}}$ it also reduces $\frac{1}{\left((1-\alpha)-1/\bar{\bar{\Gamma}}\right)\beta}$ and thus the interesting interval (the case 2 of intermediate risk) gets smaller. Finally, positive population growth reduces the sensitivity of $\check{\Gamma}$ with respect to increasing risk.

J General Intertemporal Elasticity of Substitution ρ and Risk Aversion σ

In this appendix we provide the detailed analysis of a more general utility function with intertemporal elasticity of substitution ρ and risk aversion σ summarized in Section 6.3.6 of the main text. Most of the analysis focuses on *steady states*, but we establish that our closed form results for the entire *transition* go through unchanged for an IES $\rho = 1$. We first characterize the competitive equilibrium for a given tax policy, prior to stating and analyzing the Ramsey problem.

J.1 Competitive Equilibrium for Given Tax Policy

The Euler equation with the more general utility function now reads as:

$$1 = \beta(1-\tau_{t+1})R_{t+1}\left[\int\left(\frac{c^o_{t+1}(\eta_{t+1})}{c^y_t}\right)^{1-\sigma} d\Psi(\eta_{t+1})\right]^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \int\left(\frac{c^o_{t+1}(\eta_{t+1})}{c^y_t}\right)^{-\sigma} d\Psi(\eta_{t+1}).$$

and, using the expressions for consumption in both periods and the law of motion of the capital stock, as in the previous analysis we can rewrite the first-order condition as

$$1 \;=\; \alpha\beta\left((1-\kappa)(1-\alpha)\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}(1-\tau_{t+1})k_t^{\alpha(\alpha-1)\left(1-\frac{1}{\rho}\right)} {s_t}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s_t}{s_t}\right)^{\frac{1}{\rho}}\tilde{\Gamma}.$$

In steady state the Euler equation reads as

$$1 = \alpha\beta\left((1-\kappa)(1-\alpha)\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}(1-\tau)k^{\alpha(\alpha-1)\left(1-\frac{1}{\rho}\right)}s^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s}{s}\right)^{\frac{1}{\rho}}\tilde{\Gamma}$$

where

$$k = \left[(1-\kappa)(1-\alpha)s\right]^{\frac{1}{1-\alpha}} \tag{76}$$

is the steady state capital stock and the constant $\tilde{\Gamma}$ is given by

$$\tilde{\Gamma} = v^{\left(\sigma-\frac{1}{\rho}\right)}\int(\kappa\eta_{t+1}(1-\alpha)+\alpha)^{-\sigma}\,d\Psi(\eta_{t+1}) \tag{77}$$

and v is the certainty equivalent of η defined as

$$v = \begin{cases} \left[\int(\alpha+(1-\alpha)\kappa\eta)^{1-\sigma}\,d\Psi(\eta)\right]^{\frac{1}{1-\sigma}} & \text{for } \sigma = 1 \\ \exp\left(\int\ln(\alpha+(1-\alpha)\kappa\eta)\,d\Psi(\eta)\right) & \text{otherwise.} \end{cases} \tag{78}$$

Inserting the steady state capital from equation (76) into the Euler equation delivers

$$1 = (1-\tau)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s)^{\frac{1}{\rho}}}{s}\tilde{\Gamma}. \tag{79}$$

This result is the generalization of the log-case where $\rho = \sigma = 1$, and where the steady state Euler equation was given as

$$1 \;=\; (1-\tau)\alpha\beta\left(\frac{1-s}{s}\right)\Gamma$$

Thus our previous analysis for log-utility is just a special case. Also note that if $\rho = 1$ but $\sigma \neq 1$, then the steady state Euler equation is given by

$$1 = (1-\tau)\alpha\beta\left(\frac{1-s}{s}\right)\tilde{\Gamma}$$

but the risk factor $\tilde{\Gamma}$ with $\sigma \neq 1$ differs from the risk factor Γ with $\sigma = 1$.

$$\tilde{\Gamma} = \frac{\int(\alpha+(1-\alpha)\kappa\eta)^{-\sigma}\,d\Psi(\eta)}{\int(\alpha+(1-\alpha)\kappa\eta)^{1-\sigma}\,d\Psi(\eta)} \neq \int(\kappa\eta(1-\alpha)+\alpha)^{-1}\,d\Psi(\eta) = \Gamma$$

J.1.1 Precautionary Savings Behavior in the Competitive Equilibrium

In order to aid with the interpretation of the optimal Ramsey tax rate it is useful to establish conditions under which, for a fixed tax rate, the saving rate in competitive equilibrium is increasing in income risk.

Proposition 19. *If $\tilde{\Gamma}$ is strictly increasing in income risk, then for any given tax rate $\tau \in (-\infty, 1)$ the steady state saving rate $s^{CE}(\tau)$ in competitive equilibrium is strictly increasing in income risk. If $\tilde{\Gamma}$ is strictly decreasing in income risk, then so is $s^{CE}(\tau)$.*

Proof. Rewrite equation (79) as

$$f(s) = (1-\tau)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)} \frac{(1-s)^{\frac{1}{\rho}}}{s} - \frac{1}{\tilde{\Gamma}}.$$

Then a saving rate $s^{CE}(\tau)$ that satisfies $f(s^{CE}(\tau)) = 0$ is a steady state equilibrium saving rate. We readily observe that f is continuous and strictly decreasing in s, with

$$\begin{aligned} \lim_{s \to 0} f(s) &= \infty \\ f(1) &= -\frac{1}{\tilde{\Gamma}} < 0 \end{aligned}$$

and thus for each $\tau \in (-\infty, 1)$ there is a unique $s = s^{CE}(\tau)$ that satisfies $f(s^{CE}(\tau)) = 0$. Inspection of f immediately reveals that $s^{CE}(\tau)$ is strictly increasing in $\tilde{\Gamma}$, from which the comparative statics results follow. □

Corollary 6. *For any given $\tau \in (-\infty, 1)$, the steady state saving rate $s^{CE}(\tau)$ increases in income risk if either $\rho \leq 1$, or $1 < \rho < \frac{1}{\sigma}$.*

Proof. Follows directly from the previous proposition and Lemma 2 in the main text (and proved in the next section) characterizing the behavior of $\tilde{\Gamma}$ with respect to income risk. □

Proposition 19 establishes a sufficient condition for the private saving rate to increase in income risk. But, for $\rho > \frac{1}{\sigma} > 1$ it is possible that the combination of individual savings behavior and general equilibrium factor price movements lead to the result that, for *fixed* government policy, the equilibrium saving rate is decreasing in income risk.[13] We will

[13] Also observe that a parameter constellation $1 < \rho < \frac{1}{\sigma}$ pairs a high IES with a preference for a *late* resolution of risk in a multi-period (more than two periods) model. Interestingly, the competitive equilibrium saving rate may therefore decrease in income risk precisely when we pair a high IES with a preference constellation for *early* resolution of risk.

show below that this in turn is a necessary condition for the optimal Ramsey tax rate to decrease in income risk.

J.2 The Ramsey Problem

As in Section 4, equation (12), we can write lifetime utility of a generation born in period t, in general equilibrium, as a function of the beginning of the period capital stock k_t and the saving rate s_t chosen by the Ramsey government and implemented by the appropriate choice of the capital tax τ_{t+1}.

J.2.1 Ramsey Problem for Unit IES

Now we use the formulation of lifetime utility in equation (25). Then it is straightforward to show that for $\rho = 1$ the analysis of the Ramsey problem proceeds exactly as for log utility ($\rho = \sigma = 1$), by making the problem recursive and using the method of undetermined coefficients:

$$
\begin{aligned}
W(k) &= \Theta_0 + \Theta_1 \ln(k) \\
&= \max_{s\in[0,1]} \Big\{ \ln((1-s)(1-\kappa)(1-\alpha)k^{\alpha}) \\
&\quad + \frac{\beta}{1-\sigma} \ln \int \left(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha}\right)^{1-\sigma} d\Psi(\eta) + \theta W(k') \Big\} \\
&= \max_{s\in[0,1]} \Big\{ \ln((1-s)(1-\kappa)(1-\alpha)k^{\alpha}) \\
&\quad + \frac{\beta}{1-\sigma} \ln \int \left([\kappa\eta(1-\alpha)+\alpha]\left[s(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha}\right)^{1-\sigma} d\Psi(\eta) + \theta W(s(1-\kappa)(1-\alpha)k^{\alpha}) \Big\} \\
&= \alpha\left[1+\theta\Theta_1+\alpha\beta\right]\ln(k) + \ln\left[(1-\kappa)(1-\alpha)\right] + \theta\Theta_0 + \theta\Theta_1 \ln((1-\kappa)(1-\alpha)) \\
&\quad + \beta\alpha \ln\left[(1-\kappa)(1-\alpha)\right] + \frac{\beta \ln \int \left[\kappa\eta(1-\alpha)+\alpha\right]^{1-\sigma} d\Psi(\eta)}{1-\sigma} \\
&\quad + \max_{s\in[0,1]} \left\{ \ln(1-s) + \alpha\beta \ln(s) + \theta\Theta_1 \ln(s) \right\}
\end{aligned}
$$

As in Appendix B.2, comparing the terms involving k gives the constant $\Theta_1 = \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)}$, and taking the first order condition with respect to s and solving it delivers the optimal saving rate as stated in the main text:

$$s = \frac{\alpha(\beta+\theta)}{1+\alpha\beta}.$$

This result clarifies that the closed form solution, and the fact that the optimal saving rate is constant over time and independent of the level of capital, is driven by the assumption that $IES = \rho = 1$ (and obtained for arbitrary risk aversion), whereas the size of the capital tax needed to implement the optimal Ramsey allocation evidently does depend on risk aversion σ, see Section J.1.

J.2.2 Steady State Analysis of Ramsey Problem for Arbitrary IES $\rho \neq 1$

The Ramsey government maximizing steady state lifetime utility has the objective function:

$$
\begin{aligned}
V(s) &= \frac{(c_t^y)^{1-\frac{1}{\rho}} + \beta\left\{\left[\int c_{t+1}^o(\eta_{t+1})^{1-\sigma}d\Psi\right]^{\frac{1}{1-\sigma}}\right\}^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}} \\
&= \frac{((1-\kappa)(1-s)(1-\alpha)k^\alpha)^{1-\frac{‘1}{\rho}}}{1-\frac{1}{\rho}} \\
&\quad + \frac{\beta\left[s(1-\kappa)(1-\alpha)k^\alpha\right]^{\alpha\left(1-\frac{1}{\rho}\right)}\left\{\left[\int\left\{[\kappa\eta(1-\alpha)+\alpha]\right\}^{1-\sigma}d\Psi\right]^{\frac{1}{1-\sigma}}\right\}^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}} \\
&= \frac{((1-\kappa)(1-\alpha))^{1-\frac{‘1}{\rho}}}{1-\frac{1}{\rho}}(1-s)^{\left(1-\frac{1}{\rho}\right)}k^{\alpha\left(1-\frac{1}{\rho}\right)} + \frac{\beta\left[(1-\kappa)(1-\alpha)\right]^{\alpha\left(1-\frac{1}{\rho}\right)}\tilde{\Gamma}_2}{1-\frac{1}{\rho}}s^{\alpha\left(1-\frac{1}{\rho}\right)}k^{\alpha^2\left(1-\frac{1}{\rho}\right)}
\end{aligned}
$$

where

$$
\tilde{\Gamma}_2 = \left[\int\left\{[\kappa\eta(1-\alpha)+\alpha]\right\}^{1-\sigma}d\Psi\right]^{\frac{1-\frac{1}{\rho}}{1-\sigma}}.
$$

Exploiting that in steady state

$$
k = ((1-\kappa)(1-\alpha)s)^{\frac{1}{1-\alpha}}
$$

yields

$$
\begin{aligned}
V(s) &= \frac{((1-\kappa)(1-\alpha))^{1-\frac{‘1}{\rho}}}{1-\frac{1}{\rho}}(1-s)^{\left(1-\frac{1}{\rho}\right)}((1-\kappa)(1-\alpha)s)^{\frac{\alpha\left(1-\frac{1}{\rho}\right)}{1-\alpha}} \\
&\quad + \frac{\beta\left[(1-\kappa)(1-\alpha)\right]^{\alpha\left(1-\frac{1}{\rho}\right)}\tilde{\Gamma}_2}{1-\frac{1}{\rho}}(s)^{\alpha\left(1-\frac{1}{\rho}\right)}((1-\kappa)(1-\alpha)s)^{\frac{\alpha^2\left(1-\frac{1}{\rho}\right)}{1-\alpha}} \\
&= \tilde{\phi}\left((1-s)^{\left(1-\frac{1}{\rho}\right)} + \beta\tilde{\zeta}\tilde{\Gamma}_2\right)s^{\frac{\alpha\left(1-\frac{1}{\rho}\right)}{1-\alpha}},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\phi} &= \frac{((1-\kappa)(1-\alpha))^{\frac{1-\frac{1}{\rho}}{1-\alpha}}}{1-\frac{1}{\rho}} \\
\tilde{\zeta} &= \left(\frac{1}{(1-\kappa)(1-\alpha)}\right)^{(1-\frac{1}{\rho})} > 0 \\
\tilde{\Gamma}_2 &= \left(\left[\int \{[\kappa\eta(1-\alpha)+\alpha]\}^{1-\sigma} d\Psi\right]^{\frac{1}{1-\sigma}}\right)^{1-\frac{1}{\rho}} > 0.
\end{aligned}
$$

Thus the steady state analysis in the main text carries through to Epstein-Zin-Weil utility mostly unchanged, but with the constant that maps earnings risk into the optimal saving rate now being affected both by risk aversion and the IES.

Taking the first order condition, the optimal steady state saving rate is defined implicitly by the equation

$$\frac{s}{(1-s)^{\frac{1}{\rho}}} = \frac{\alpha}{1-\alpha}(1-s)^{\left(1-\frac{1}{\rho}\right)} + \beta\frac{\alpha}{1-\alpha}\tilde{\zeta}\tilde{\Gamma}_2 \tag{80}$$

and rewriting this equation yields

$$LHS(s) = s = \frac{\alpha}{1-\alpha}\left[(1-s) + \beta\tilde{\zeta}\tilde{\Gamma}_2(1-s)^{\frac{1}{\rho}}\right] = RHS(s). \tag{81}$$

We observe that the left hand side is linearly increasing in s, with $LHS(0) = 0$ and $LHS(1) = 1$ and the right hand side is strictly decreasing in s, with $RHS(0) > 0$ and $RHS(1) = 0$. Since both sides are continuous in s, from the intermediate value theorem it follows that there is a unique $s^* \in (0,1)$ solving the first order condition of the Ramsey problem (81). Since $RHS(s)$ is strictly increasing in $\tilde{\Gamma}_2$, the Ramsey saving rate is strictly increasing in $\tilde{\Gamma}_2$. We then have

Proposition 20. *Suppose that $\theta = 1$ and thus the Ramsey government maximizes steady state welfare. There exists a unique optimal Ramsey saving rate $s^* \in (0,1)$ solving equation (81). This saving rate is strictly increasing in the risk constant $\tilde{\Gamma}_2$ and can be implemented with a capital tax rate τ^* determined by the competitive equilibrium Euler equation:*

$$1 = (1-\tau^*)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s^*)^{\frac{1}{\rho}}}{s^*}\tilde{\Gamma}. \tag{82}$$

For future reference we rewrite equation (81) as

$$\frac{(1-s)^{\frac{1}{\rho}}}{s} = \frac{\frac{1-\alpha}{\alpha} - \frac{(1-s)}{s}}{\beta\tilde{\zeta}\tilde{\Gamma}_2} = \frac{\frac{1}{\alpha} - \frac{1}{s}}{\beta\tilde{\zeta}\tilde{\Gamma}_2}. \tag{83}$$

J.2.3 Comparative Statics with Respect to Income Risk

In Appendix K.1 we prove the following result relating the extent of income risk to the constants $\tilde{\Gamma}, \tilde{\Gamma}_2$ which are in turn crucial for determining comparative statics results.

Lemma 2. *An increase in income risk (a mean-preserving spread of η) increases $\tilde{\Gamma}_2$ if and only if $\rho \leq 1$ and increases $\tilde{\Gamma}$ if $\rho \leq 1$, or $\rho > 1$ and $\sigma < 1/\rho$.*

Note that the condition that characterizes the relation between income risk and $\tilde{\Gamma}_2$ is necessary and sufficient whereas the two alternative conditions that characterize the relation between income risk and $\tilde{\Gamma}$ are only sufficient.[14] We provide further intuition for this result below when discussing implementation of the optimal Ramsey policy. We now derive the comparative statics of s^* and τ^* with respect to income risk discussed in the main text.

Risk and the Optimal Saving Rate The comparative static results of the steady state Ramsey saving rate with respect to income risk is stated in the next:

Proposition 21. *An increase in income risk increases the optimal steady state Ramsey saving rate s^* if and only if $\rho < 1$ and decreases it if and only if $\rho > 1$.*

The proof of this result follows directly from Lemma 2 and Proposition 20. Thus the direction of the change in s with respect to income risk is exclusively determined by the IES ρ, with the log-case acting as a watershed. Of course how strongly the saving rate responds to an increase in income risk is also controlled by risk aversion through the term $\tilde{\Gamma}_2$. What is the intuition for this result? Suppose the economy is in the steady state associated with a given extent of income risk and the optimal Ramsey tax policy, and now consider an increase in income risk. The Ramsey government can always neutralize the response of private households' savings behavior, by appropriate adjustment of the tax rate on capital to implement the new optimal saving rate.[15]

[14]The sufficient conditions provided in the Lemma are stated in the Propositions 5 and 6 of Kimball and Weil (2009).

[15]We saw this explicitly in the decomposition of the first order condition of the Ramsey government in Section 4.1, where the risk term Γ from the competitive equilibrium optimality condition dropped out because

The question is then how the saving rate desired by the Ramsey government itself changes. Households (and thus the Ramsey government) obtain utility from safe consumption when young and risky consumption when old, and the desire for smoothing utility from safe consumption when young and the certainty equivalent of consumption when old is determined by the IES ρ. As risk increases, old age consumption is now a less effective way to generate utility, and the certainty equivalent of old-age consumption declines, holding the consumption allocation constant. Whether the Ramsey government wants to raise or lower old-age consumption (by increasing or reducing the saving rate) depends on how much households value a smooth life cycle utility profile. In the log-case the two forces exactly balance out and the Ramsey saving rate does not respond to income risk at all. In contrast, if households strongly desire a smooth path of (the certainty equivalent of) consumption, then the Ramsey government compensates for the loss of old-age certainty equivalent consumption from larger income risk by saving at a higher rate, and s increases with income risk if the IES ρ is small. The reverse is true for a high IES.

Risk and the Optimal Tax Rate Finally, we can also determine the impact of income risk on optimal steady state capital taxes. From equation (82) the optimal Ramsey tax rate is given by

$$1 = (1-\tau^*)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)} \frac{(1-s^*)^{\frac{1}{\rho}}}{s^*}\tilde{\Gamma}. \tag{84}$$

We observe that income risk affects the optimal tax rate in two ways. First, for a given target saving rate s^*, the direct impact of income risk depends on how $\tilde{\Gamma}$ (and thus the private saving rate) responds to an increase in risk. Second, a change in income risk changes the optimal saving rate s^* through $\tilde{\Gamma}_2$, as characterized in the previous proposition.

Proposition 22. *If $\rho \leq 1$, then an increase in income risk increases the optimal tax rate on capital. Similarly, if $\rho > 1$ and $\sigma \leq 1/\rho$, then an increase in income risk increases the optimal tax rate on capital. If $\rho > 1$ and $\sigma > 1/\rho$, an increase in income risk might lead to a strict reduction in the optimal tax rate τ on capital. A necessary condition for this to occur is that the competitive equilibrium saving rate for given τ is strictly decreasing in income risk.*

The intuition for the last part of the proposition is that, if $\rho > \max\{1, 1/\sigma\}$, then private

the government chooses, through taxes and the associated changes in factor prices, to exactly offset the impact of higher risk on private household savings decisions. In the logic of that section, an increase in Γ increases $PE(s)$ but reduces $GE(s)$ by precisely the same factor.

households might *decrease* their saving rate too much in general equilibrium in response to an increase in income risk since they do no internalize the impact of the decline of the saving rate on the capital stock and thus on wages of future generations. For the capital tax to decrease in income risk this future generations effect has to be sufficiently strong. To see this formally, in the next paragraph we first derive the decomposition of the first-order condition for the optimal saving rate into the terms $PE(s), GE(s)$ and $FG(s)$ for the general EZW utility function, and then we use this decomposition to write equation (84) as

$$1 = \underbrace{(1-\tau^*)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } PE(s)+GE(s)} \underbrace{-(1-\tau^*)\frac{\alpha}{s^*}\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } FG(s)}.$$

Since $\tilde{\Gamma}/\tilde{\Gamma}_2$ is increasing in income risk, the optimal capital tax rate τ^* can only decrease in income risk when the last term, the future generations effect, is large. This effect calls for a tax rate that decreases with income risk since s^* is decreasing in risk for $\rho > 1$.

J.3 Details of Proposition 22

J.3.1 Implementation

We start with a discussion of the optimal tax rate in the steady state. The optimal steady state capital tax rate τ^* satisfies, from equation (79)

$$1 = (1-\tau^*)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s^*)^{\frac{1}{\rho}}}{s^*}\tilde{\Gamma}. \tag{85}$$

We observe that the optimal tax rate is strictly increasing in $\tilde{\Gamma}$ and strictly decreasing in the Ramsey saving rate s^* that is to be implemented. Further, recall that the Ramsey saving rate s^* itself satisfies the first order condition (83)

$$\frac{(1-s^*)^{\frac{1}{\rho}}}{s^*} = \frac{\frac{1}{\alpha}-\frac{1}{s^*}}{\beta\tilde{\zeta}\tilde{\Gamma}_2} \tag{86}$$

and is impacted by income risk through $\tilde{\Gamma}_2$. Plugging (86) into (85) and exploiting the definition of $\tilde{\zeta}$ yields

$$1 = (1-\tau^*)\left(1-\frac{\alpha}{s^*}\right)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}. \tag{87}$$

Lemma 2 establishes that $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is strictly increasing in income risk, and Proposition 21 in the main text establishes that an increase in income risk increases s^* if and only if $\rho < 1$ and decreases it if and only if $\rho > 1$. To sign the overall impact of income risk on the capital tax rate it is therefore useful to consider the following cases:

Case $\rho \leq 1$. This case gives clean results. From equation (87), since $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is strictly increasing in income risk, and since s^* is increasing in income risk for $\rho \leq 1$, strictly so if $\rho < 1$, it follows that τ^* is strictly increasing in risk.

Case $\rho > 1$ and $\sigma \leq 1/\rho$. In this case $\tilde{\Gamma}$ is strictly increasing in risk (Lemma 2) and s^* is strictly decreasing in risk (see Proposition 21). It then directly follows from equation (87) that τ^* is strictly increasing in income risk as well.

Case $\rho > 1$ and $\sigma > 1/\rho$. Since $\rho > 1$, the Ramsey saving rate s^* is strictly decreasing in income risk (which by itself calls for a tax rate that is strictly increasing in income risk), by equation (85). However, now the direct impact of income risk on taxes through the term $\tilde{\Gamma}$ might call for lower taxes since $\tilde{\Gamma}$ might now be decreasing in income risk. If $\tilde{\Gamma}$ is weakly increasing in income risk, then so is τ^*. Thus a necessary condition for τ^* to decrease with income risk is for $\tilde{\Gamma}$ to be strictly decreasing with income risk. This in turn is a necessary and sufficient condition for the private saving rate in competitive equilibrium to decrease with income risk (see Proposition 19). Thus the Ramsey tax rate τ^* is strictly decreasing in income risk *only if* the private saving rate $s^{CE}(\tau)$ is strictly decreasing in income risk (for any given tax rate τ).

Finally, one might conjecture that, since $\rho > 1$ and $\sigma > 1/\rho$ is required for the capital tax to decrease in income risk, that as long as both parameters are large enough the result will materialize. This conjecture turns out to be false, as an investigation of the most extreme case $\rho = \sigma = \infty$ shows. In this case lifetime utility is given by

$$V_t = c_t^y + \beta \underline{c}_{t+1}^o \tag{88}$$

where $\underline{c}_{t+1}^o$ is consumption in old age if the lowest possible labor productivity realization $\eta = \underline{\eta}_{t+1}$ materializes. In this case one can solve analytically for the optimal interior Ramsey saving and tax rate, and show that the optimal tax rate is the higher the lower is

$\underline{\eta}_{t+1}$ and thus the higher is income risk.[16]

Proposition 4 above provides a fairly general implementation result for expected utility along the transition. The next proposition extends this result to EZW utility.

Proposition 23. *If the utility function is of the EZW form, then in general equilibrium we have* $s_\tau = \frac{\partial s_t}{\partial \tau_{t+1}} < 0$ *and unambiguous implementation.*

Proof. Recall from Section J.1 that the first-order condition in any period t of the transition is

$$1 = \alpha\beta\left((1-\kappa)(1-\alpha)\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}(1-\tau_{t+1})k_t^{\alpha(\alpha-1)\left(1-\frac{1}{\rho}\right)}{s_t}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s_t}{s_t}\right)^{\frac{1}{\rho}}\tilde{\Gamma}.$$

Observe that an increase in the tax rate decreases the RHS. Collect terms on the saving rate as

$${s_t}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s_t}{s_t}\right)^{\frac{1}{\rho}} = {s_t}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)-\frac{1}{\rho}}\left(1-s_t\right)^{\frac{1}{\rho}}$$

and notice that for any $\rho > 0$ term $(1-s_t)^{\frac{1}{\rho}}$ decreases in the saving rate. In response to an increase of the tax rate this force drives the saving rate down. To get unambiguous implementation, we thus require that the exponent

$$(\alpha-1)\left(1-\frac{1}{\rho}\right)-\frac{1}{\rho}<0 \quad\Leftrightarrow\quad \frac{1}{\rho}>0>1-\frac{1}{\alpha}$$

which holds for all $\alpha \in (0,1)$. □

J.3.2 Decomposition of the FOC into $PE(s)$, $GE(s)$ and $FG(s)$

Now we decompose the first order condition of the Ramsey problem into three terms:

[16]In this case it is possible that the Ramsey government will want to implement a saving rate of $s = 1$ since households have linear preferences over consumption when young and minimum (across η) consumption when old. As long as $\underline{\eta}$ is sufficiently small, however, the Ramsey government prefers to implement an interior saving rate.

Proposition 24. *For $\theta = 1$, $\sigma \neq \frac{1}{\rho}$, terms $PE(s), GE(s), FG(s)$ are given by*

$$
\begin{aligned}
PE(s) &= -\frac{1}{1-s}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s}\tilde{\Gamma}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)} \\
GE(s) &= \frac{\alpha\beta}{s}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\left(\tilde{\Gamma}_2 - \tilde{\Gamma}\right) \\
FG(s) &= \frac{\alpha}{s(1-\alpha)}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha^2\beta}{s(1-\alpha)}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\tilde{\Gamma}_2
\end{aligned}
$$

where $k(s) = (s(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}$ is the steady state capital stock.

Therefore,

$$
PE(s) + GE(s) = -\frac{1}{1-s}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\tilde{\Gamma}_2. \tag{89}
$$

and

$$
PE(s)+GE(s)+FG(s) = \left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}}\left(\frac{\alpha}{s(1-\alpha)} - \frac{1}{1-s}\right) + \frac{1}{s}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\frac{\alpha\beta}{(1-\alpha)}\tilde{\Gamma}_2. \tag{90}
$$

Thus, compared to the expressions for these three effects we derived in Section 4.1, the partial equilibrium precautionary savings effect still cancels out the current generations general equilibrium effect ($\tilde{\Gamma}$ cancels out when adding up $PE(s)$ and $GE(s)$). However, additionally risk enters through $\tilde{\Gamma}_2$. With $\rho < 1$ an increase of risk increases $\tilde{\Gamma}_2$ thereby pushing up the desired saving rate of the Ramsey planner. The reason is that an increase of risk decreases the utility value of second period consumption of current generations (effect in $GE(s)$) and of all future generations (effect in $FG(s)$). With a low IES, it is optimal to compensate this with higher savings; vice versa for a high IES where the Ramsey planner rather prefers increased first-period consumption, respectively current generations consumption, over future consumption in response to an increase in risk.

Proof of Proposition 24. Calculating the respective terms yields

$$PE(s) = (1-\kappa)(1-\alpha)k^{\alpha}\Big[-\left((1-s)(1-\kappa)\left(1-\alpha\right)k^{\alpha}\right)^{-\frac{1}{\rho}} +$$

$$\alpha k'(s)^{\alpha-1}\beta\left(\int\left(\kappa\eta(1-\alpha)+\alpha\right)^{1-\sigma}d\Psi\right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} k'(s)^{\alpha\left(\sigma-\frac{1}{\rho}\right)}\int\left(\kappa\eta(1-\alpha)+\alpha\right)^{-\sigma}d\Psi k'(s)^{-\alpha\sigma}\Bigg]$$

$$= -\frac{1}{1-s}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s}\tilde{\Gamma}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}.$$

and for

$$GE(s) = \beta\left(\int c^{o}(\eta)^{1-\sigma}d\Psi\right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\int\left(c^{o}(\eta)^{-\sigma}\right)\left[\kappa\eta w'(s)+(1-\kappa)(1-\alpha)k^{\alpha}R'(s)s\right]d\Psi(\eta)$$

$$= \beta\Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} k'(s)^{\alpha\left(\sigma-\frac{1}{\rho}\right)}\int\left(\kappa\eta(1-\alpha)+\alpha\right)^{-\sigma}k'(s)^{-\alpha\sigma}\alpha(1-\alpha)s^{-1}$$

$$\cdot\left[\kappa\eta k'(s)^{\alpha}-(1-\kappa)(1-\alpha)k^{\alpha}k'(s)^{\alpha-1}s\right]d\Psi$$

$$= \frac{\alpha\beta}{s}k'(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\int\left(\kappa\eta(1-\alpha)+\alpha\right)^{-\sigma}\left[\kappa\eta(1-\alpha)+\alpha-1\right]d\Psi$$

$$= \frac{\alpha\beta}{s}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\left(\tilde{\Gamma}_2-\tilde{\Gamma}\right).$$

When maximizing steady state utility, $FG(s)$ is equivalent to the derivative of the utility function with respect to the current period capital stock. Therefore:

$$FG(s) = u_{c^y}c^y_{k(s)}k(s)_s + \beta\left(\int c^{o}(\eta)^{1-\sigma}d\Psi\right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\int\left(c^{o}(\eta)^{-\sigma}\right)c^{o}_{k'(s)}k'(s)_{k(s)}k(s)_s d\Psi,$$

where

$$u_{c^y}c^y_{k(s)}k(s)_s = \left((1-s)(1-\kappa)(1-\alpha)k(s)^{\alpha}\right)^{-\frac{1}{\rho}}(1-s)(1-\kappa)(1-\alpha)\alpha k(s)^{\alpha-1}(1-\kappa)k(s)^{\alpha}$$

$$= \frac{\alpha}{s(1-\alpha)}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k'(s)^{1-\frac{1}{\rho}}$$

and

$$\left(\int c^{o}(\eta)^{1-\sigma}d\Psi\right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} = \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} k'(s)^{\alpha\left(\sigma-\frac{1}{\rho}\right)}$$

and

$$\beta \int c^{0-\sigma} c^o_{k'(s)} k'(s)_{k(s)} k(s)_s d\Psi =$$

$$\beta \int \left(\kappa\eta(1-\alpha)+\alpha\right)^{-\sigma} k'(s)^{-\alpha\sigma} \left(\kappa\eta(1-\alpha)+\alpha\right) d\Psi \alpha k'(s)^{\alpha-1} k'(s)\alpha(1-\kappa)k(s)^{\alpha-1}$$

$$= \frac{\alpha^2\beta}{s(1-\alpha)} k'(s)^{\alpha(1-\sigma)}\Gamma_2.$$

Therefore:

$$FG(s) = \frac{\alpha}{s(1-\alpha)} \left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha^2\beta}{s(1-\alpha)} k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\tilde{\Gamma}_2.$$

□

J.3.3 Decomposition of τ^*

Given this result, the optimal tax rate τ^* can be decomposed as stated in the previous paragraph:

Corollary 7. *τ^* can only be decreasing in risk if the effect of $FG(s)$ is sufficiently strong.*

Proof. We know that the FOC for s^* follows from

$$PE(s) + GE(s) + FG(s) = 0$$

Now set $FG(s) = 0$. Rewrite from (89)

$$PE(s) + GE(s) = 0 \;\Leftrightarrow\; \frac{s}{(1-s)^{\frac{1}{\rho}}} = \alpha\beta\tilde{\zeta}\tilde{\Gamma}_2,$$

which uses $k(s) = (s(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}$ and $\tilde{\zeta} = ((1-\alpha)(1-\kappa))^{\frac{1}{\rho}-1}$. Using the above in (84) gives

$$1 = (1-\tau^*)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$$

and $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is unambiguously increasing in risk, see Section K.1. Using the above we can thus decompose equation (81) as described above:

$$1 = \underbrace{(1-\tau^*)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } PE(s)+GE(s)} \underbrace{-(1-\tau^*)\frac{\alpha}{s^*}\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } FG(s)}.$$

□

J.4 Pareto Improving Transitions

Observe that specification (11) nests EZW preferences as a special case. Thus, Proposition 3 and Corollary 3 apply.

K Income Risk and $\Gamma, \Gamma_2, \tilde{\Gamma}, \tilde{\Gamma}_2$

K.1 General Case

In this section we prove Lemma 2 in Subsection J.2.3 through two separate Lemmas. For this, recall that the relevant expressions involving idiosyncratic income risk are given by:

$$\begin{aligned}
\Gamma &= \int (\kappa\eta(1-\alpha)+\alpha)^{-\sigma}\, d\Psi(\eta) \\
\Gamma_2 &= \int (\kappa\eta(1-\alpha)+\alpha)^{1-\sigma}\, d\Psi(\eta) \\
\tilde{\Gamma} &= \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\Gamma = v^{\sigma-\frac{1}{\rho}}\Gamma \\
\tilde{\Gamma}_2 &= \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\Gamma_2 = \Gamma_2^{\frac{1-\frac{1}{\rho}}{1-\sigma}} = v^{1-\frac{1}{\rho}} \\
\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2} &= \frac{\Gamma}{\Gamma_2} \\
v &\equiv \begin{cases} \left[\int (\alpha+(1-\alpha)\kappa\eta)^{1-\sigma}\, d\Psi(\eta)\right]^{\frac{1}{1-\sigma}} & \text{for } \sigma \neq 1 \\ \exp\left[\int \ln(\alpha+(1-\alpha)\kappa\eta)\, d\Psi(\eta)\right] & \text{for } \sigma = 1 \end{cases}
\end{aligned}$$

Furthermore, as in the main text we use the notion of a mean-preserving spread in the random variable η when referring to an increase in risk, that is, formally, random variable η is replaced by $\tilde{\eta} = \eta + \nu$, where ν is a random variable with zero mean and positive variance (and Assumption 1 applies to $\tilde{\eta}$ as well).

Lemma 3. *The certainty equivalent v is decreasing in η-risk.*

Proof. If $\sigma > 1$ ($\sigma < 1$), then $(\alpha + (1-\alpha)\kappa\eta)^{1-\sigma}$ is convex and downward sloping (concave and upward sloping) in η. The certainty equivalent of a convex and downward sloping (respectively, concave and upward sloping) function is decreasing in risk. $\square$

Lemma 4. *The comparative statics of the other risk terms with respect to a mean-preserving spread in η are given by:*

1. *Γ is increasing in η-risk.*
2. *Γ_2 is increasing (respectively, decreasing) in η-risk if $\sigma > 1$ (respectively $\sigma < 1$).*
3. *$\tilde{\Gamma}_2$ is increasing (decreasing) in η-risk if $\rho < 1$ ($\rho > 1$).*
4. *For $\rho < 1$, $\tilde{\Gamma}$ is increasing in η-risk. For $\rho > 1$ we have the following case distinction:*
   (a) For $\frac{1}{\sigma} > \rho > 1$, $\tilde{\Gamma}$ unambiguously increases in income risk.
   (b) For $\rho > 1, \rho > \frac{1}{\sigma} > 0$, i.e., $\sigma < \infty$ the effect of η-risk on $\tilde{\Gamma}$ is ambiguous.

Proof.

1. Γ is increasing in η-risk because $(\kappa\eta(1-\alpha) + \alpha)^{-\sigma}$ is a convex function in η (with the degree of convexity increasing in σ).
2. Γ_2 is increasing (decreasing) in η-risk if $\sigma > 1$ ($\sigma < 1$) because $(\kappa\eta(1-\alpha) + \alpha)^{1-\sigma}$ is a convex (concave) function of η.
3. $\tilde{\Gamma}_2$ is increasing (decreasing) in η-risk if $\rho < 1$ ($\rho > 1$) because the certainty equivalent v decreases in η-risk and because for $\rho < 1$ ($\rho > 1$) the exponent $1 - \frac{1}{\rho}$ is negative (positive).
4. For $\rho < 1$, $\tilde{\Gamma}$ is increasing in η-risk (sufficient condition). To see this, rewrite $\tilde{\Gamma}$ as

$$\tilde{\Gamma} = \frac{\Gamma}{\Gamma_2^{-\frac{-(1-\sigma)+(1-\frac{1}{\rho})}{1-\sigma}}} = \frac{\Gamma}{\Gamma_2^{1-\frac{1-\frac{1}{\rho}}{1-\sigma}}} = \frac{\Gamma}{\Gamma_2}\Gamma_2^{\frac{1-\frac{1}{\rho}}{1-\sigma}} = \frac{\Gamma}{\Gamma_2}v^{1-\frac{1}{\rho}} \tag{91}$$

Notice that for $\sigma \leq 1$, $\frac{\Gamma}{\Gamma_2}$ is the ratio of the expectation of a strictly convex and a concave function. Hence, for $\sigma \leq 1$ the term $\frac{\Gamma}{\Gamma_2}$ is increasing in risk by Jensen's inequality. For $\sigma > 1$ term $\frac{\Gamma}{\Gamma_2}$ is the ratio of the expectation of two convex functions with the convexity of the function in the numerator, $(\kappa\eta(1-\alpha)+\alpha)^{-\sigma}$, being stronger than in the denominator, $(\kappa\eta(1-\alpha)+\alpha)^{1-\sigma}$ as long as $\sigma<\infty$. Therefore, also for $1<\sigma<\infty$ term $\frac{\Gamma}{\Gamma_2}$ is increasing in risk. For $\sigma=\infty$ term $\frac{\Gamma}{\Gamma_2}$ is equal to 1. Finally, since the certainty equivalent v is decreasing in η-risk, term $v^{1-\frac{1}{\rho}}$ increases in η-risk if and only if $\rho<1$.

5. For $\rho>1$ we have the following case distinction, based on the representation of $\tilde{\Gamma}=v^{\sigma-\frac{1}{\rho}}\Gamma$:

   (a) For $\frac{1}{\sigma}>\rho>1$, $\tilde{\Gamma}$ unambiguously increases in η-risk because v decreases in η-risk and $\sigma-\frac{1}{\rho}<0$.

   (b) For $\rho>1,\rho>\frac{1}{\sigma}$ the effect of η-risk on $\tilde{\Gamma}$ is ambiguous because v is decreasing in η-risk and $\sigma-\frac{1}{\rho}>0$ so that $v^{\sigma-\frac{1}{\rho}}$ is decreasing in η-risk whereas Γ is increasing in η-risk. Rewriting $\tilde{\Gamma}$ as in equation (91) does not resolve this ambiguity because term $\frac{\Gamma}{\Gamma_2}$ is increasing in η-risk whereas $v^{1-\frac{1}{\rho}}$ is decreasing in η risk because $1-\frac{1}{\rho}>0$.

□

K.2 Expressing Γ-Intervals from Proposition 4 in Terms of Variances

The bounds in Proposition 4 can be given in terms of the variances of the income shock η, to a second-order Taylor approximation of the integral defining Γ. This approximation around $\eta=1$ gives

$$\Gamma(\alpha,\kappa,\sigma,\Psi)\approx\bar{\Gamma}+\frac{[\kappa(1-\alpha)]^2}{[\kappa(1-\alpha)+\alpha]^3}\sigma_\eta^2.$$

With this approximation the interval for intermediate risk, item 2 of Proposition 4, becomes $\sigma_\eta^2\in\left(\underline{\sigma_\eta^2},\overline{\sigma_\eta^2}\right)$ where

$$\underline{\sigma_\eta^2}=\frac{(\kappa(1-\alpha)+\alpha)^3}{(\kappa(1-\alpha))^2}\left(\frac{1+\beta}{(1-\alpha)\beta}-\bar{\Gamma}\right)$$
$$\overline{\sigma_\eta^2}=\frac{(\kappa(1-\alpha)+\alpha)^3}{(\kappa(1-\alpha))^2}\left(\frac{1}{\left((1-\alpha)-\frac{1}{\bar{\Gamma}}\right)\beta}-\bar{\Gamma}\right)$$

and $\overline{\sigma_\eta^2}>\underline{\sigma_\eta^2}>0$ under the maintained assumption that $\beta<\left[(1-\alpha)\bar{\Gamma}-1\right]^{-1}$. Thus, all intervals defined in Proposition 4 can be expressed in terms of variances and are nonempty. Also note that if the distribution Ψ is log-normal and thus exclusively determined by its variance (given that the mean is pinned down by the assumption $E(\eta) = 1$), then no second order approximation is necessary in the above argument, but the mapping between the variance bounds and the Γ bounds is algebraically much more involved.

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