Who Saves More, the Naive or the Sophisticated Agent?*

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Abstract

This paper studies discrete time finite horizon life-cycle models with arbitrary discount functions and iso-elastic per period power utility featuring resistance to inter-temporal substitution of θ. We distinguish between the savings behavior of a sophisticated versus a naive agent. Although both agent types have identical preferences, they solve different utility maximization problems whenever the model is dynamically inconsistent. Pollak (1968) shows that the savings behavior of both agent types is nevertheless identical for logarithmic utility ($\theta = 1$). We generalize this result by showing that the sophisticated agent saves in every period a greater fraction of her wealth than the naive agent if and only if $\theta \geq 1$. While this result goes through for model extensions that preserve linearity of the consumption policy function, it breaks down for non-linear model extensions.

JEL Classification: D15, D91, E21.

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1 Introduction

How time preferences and beliefs about the future affect consumption savings decisions is a classical economic question. The workhorse model of inter-temporal allocation is the life-cycle model of Modigliani and Brumberg (1954) and Ando and Modigliani (1963). Standard deterministic models assume an additively separable per period utility function, where future utility is discounted exponentially (Samuelson 1937). Models with survival beliefs express those as additive probability measures. *Standard discounting* as the combination of exponential time-discounting with additive survival beliefs result in dynamically consistent life-cycle models in which the future selves of the economic agent have no incentives to deviate from her ex ante optimal consumption and savings plan. This paper instead analyzes life-cycle models with *arbitrary* effective discount factors which are, in general, dynamically inconsistent.

We follow the literature since Strotz (1955) and Pollak (1968) and compare a naive agent—who is not aware that her future selves might have deviating preferences—with a sophisticated agent—who correctly anticipates the consumption choices of her future selves. As our main research question we investigate how the underlying effective discounting process impacts on the question whether the naive or the sophisticated agent saves a greater fraction of her wealth in any given time-period. From an economic policy perspective this question is relevant because governments worldwide look for ways to induce more prudent savings behavior. If it was the case that a sophisticated agent will always save more than a naive agent under an empirically relevant discounting scenario, awareness campaigns about people's dynamic inconsistencies may usefully complement financial incentives schemes.

Intuitively, one may expect that the nature of the effective discounting processes has some impact on the question whether a sophisticated agent saves more than her naive counterpart or vice versa. This intuition is typically based on preferences with quasi-hyperbolic time-discounting (QHD) for which discount factors take on the simple form $1, \beta\delta, \beta\delta^2, \ldots$, where $\delta > 0$ stands for the (standard) exponential long-term time-discount factor and $\beta > 0$ denotes a short-term time-discount factor. The agent exhibits a *presence bias* if $\beta < 1$ whereas she exhibits a *future bias* if $\beta > 1$. A naive agent will save 'too little' in hindsight if she has a presence bias whereas she saves 'too much' if she has a future bias. Allowing this agent to understand her bias, i.e., turning her into a sophisticated agent, might therefore suggest that a sophisticated agent with presence (resp. future) bias will save more (resp. less) than her naive counterpart. This intuition is, however, flawed as it implicitly assumes that today's sophisticated agent reacts against the savings behavior of *naive* future selves. But the decision situation of today's sophisticated agent is strategically more complex because she reacts against her *sophisticated* future selves.

We address our research question within a class of life-cycle models with a per period iso-elastic power utility function for which the consumption policy functions will be linear in wealth for our baseline cake-eating model. We denote the concavity parameter of the per period utility function by θ so that $1/\theta$ measures the inter-temporal elasticity of substitution (IES). For this preference class we find that irrespective of the effective discount process, the sophisticated agent saves more than her naive counterpart in any given time period if and only if the IES is smaller than one. We regard this finding as surprising because the effective discount process is the sole reason for why the model might be dynamically inconsistent to begin with.

Our result thereby extends Pollak (1968)'s finding for the class of preferences with iso-elastic power per period utility that for an IES of unity the naive and the sophisticated agent save exactly the same fraction of their wealth in every period irrespective of their effective discount function. We thus show that the *IES* not only measures the *strength* of the inter-temporal substitution to variations of discount factors but also the *sign* of the direction of substitution to departures from standard discounting across the two types of agents.

We start with a cake-eating baseline model with finite horizon $T < \infty$. In this model, households have age-dependent effective discount factors $\rho_{h,t}$, with $\rho_{h,t} > 0$ and $\rho_{t,t} = 1$, where h is the current age of the household and t is the age with the respective future period consumption delivery. We are agnostic about the nature of the discount process. In deterministic models it reflects pure time-discounting and in models with survival uncertainty a combination of pure time-discounting and survival beliefs.[1] Because the discount factors of the h-old agent, $h = 0, ..., T-1$, can be any strictly positive real-numbers, our life cycle model is very general and it encompasses relevant extensions of the standard model such as hyperbolic as well as quasi-hyperbolic time-discounting models which are characterized by $\rho_{h,t} = \beta\delta^{t-h}$ for $t > h$ with $\beta > 0$ denoting a short-term and $\delta > 0$ denoting a long-term time discount factor (cf. Phelps and

[1] Compare, e.g., Halevy (2008), Epper, Fehr-Duda, and Bruhin (2011), Saito (2011), Chakraborty, Halevy, and Saito (2020) who discuss the relationship between pure time-preferences and preferences under uncertainty or/and risk.

Pollak 1968; Laibson 1997; 1998; O'Donoghue and Rabin 1999; Harris and Laibson 2001) and Choquet expected utility or/and Prospect theory life-cycle models with non-additive subjective survival beliefs (cf. Bleichrodt and Eeckhoudt 2006; Ludwig and Zimper 2013; Drouhin 2015; Groneck, Ludwig, and Zimper 2016; Grevenbrock, Groneck, Ludwig, and Zimper 2021 and references therein). To make this latter point explicit, one can show that the discount factors of an h-old Choquet expected utility decision maker are $\rho_{h,t} = \delta_{h,t}\nu_{h,t}$, where $\delta_{h,t}$ stands for pure time-discounting between present age h and future age t and $\nu_{h,t}$ stands for the decision maker's non-additive belief to survive from age h to age t.[2]

We solve this baseline model for the savings behavior of the sophisticated agent and her naive counterpart, respectively. For interpretational reasons, we describe each agent type's utility maximization problem in terms of a best response function against the anticipated consumption choices of her future selves. Although both agent types share at every age the same preferences—and thereby the same best response function—, they solve different utility maximization problems whenever the model is dynamically inconsistent due to their different anticipations about future consumption choices. The naive agent chooses her per-period consumption under the anticipation that her future selves will stick to the consumption choices that would be optimal from her ex ante perspective. In contrast, the sophisticated agent correctly anticipates the consumption choices of her future selves and thus chooses her per-period consumption in accordance with a subgame-perfect Nash equilibrium.

Turning now to our main result, denote by m^i_h, for $i \in \{n, s\}$ the *marginal propensity to consume* (=MPC) out of total wealth w_h at age h of the *n*aive and the *s*ophisticated agent, respectively. We derive Theorem 1, which states that for all (arbitrary) specifications of the effective discount factors , (i) $\theta < 1$ implies $m^n_h \leq m^s_h$ and (ii) $\theta > 1$ implies $m^n_h \geq m^s_h$. If the model is dynamically inconsistent, the respective MPC inequalities are strict for all but the last two periods of life. This result is directly derived from a comparison of the respective MPCs, thus it holds globally and hinges on the linearity of the consumption policy function. In a quantitative illustration we further show that the differences in MPCs across the two types of agents can be large for plausible values of θ.

We extend our main finding to models with uninsurable return risk, a portfolio choice and homothetic Epstein-Zin-Weil (EZW) preferences (Epstein and Zin 1989; Epstein and Zin 1991;

[2]A formal proof can be found in the earlier version of this paper (Groneck, Ludwig, and Zimper 2021).

Weil 1989) nesting standard CRRA preferences as a special case, again for the multi-period model where $T \geq 2$. Our main result therefore holds in a broad class of economic models as long as the linearity of consumption policy functions in wealth is preserved. With the extension to Epstein-Zin-Weil preferences we also formally establish that it is the deviation of the IES from unity which governs the direction of the relative consumption-savings response of the two agent types to deviations from standard discounting and not risk aversion.[3]

In contrast, our main result does not extend to models with standard additive risky labor income and borrowing constraints. These model extensions have in common that they violate linearity of the consumption policy function. We provide a tentative game-theoretic interpretation for this finding. Linearity of the policy function in the baseline model means that ex ante MPC choices have no impact on future MPC choices because these future choices are independent of handed-down wealth levels. For logarithmic utility the identical savings behavior of the sophisticated agent and her naive counterpart is supported by a (subgame-perfect) Nash equilibrium in spite of the fact that the naive agent bases her choices on incorrect anticipations, which, for purely technical reasons, result in the same best responses as the correct anticipations of the sophisticated agent. In model extensions with non-linear consumption policy functions, however, the strategic situation becomes more complex because handed-down wealth levels matter for future MPC choices. Whereas the sophisticated agent's utility maximization problem captures this new strategic complexity, the naive agent remains unaware of the strategic dimension of her decision problem. Due to this added strategic complexity, it does not come as a surprise that Pollak's (1968) observational equivalence result breaks down for model extensions with non-linear consumption policy functions.

Related literature. There exists a large behavioral and decision-theoretic literature which argues that human decision making is typically prone to violations of dynamic consistency. Dynamic inconsistencies arise, for example, within the following three modeling classes: (i) deterministic models with a presence bias induced by hyperbolic or quasi-hyperbolic time-discounting

[3]In this extension, the curvature parameter θ takes (at least) a triple role as a measure of resistance to intertemporal substitution, a parameter partially controlling precautionary savings behavior (Kimball and Weil 2009; Krueger, Ludwig, Villalvazo 2021), and a parameter controlling the sign of the naive and sophisticated agents' savings response to deviations from standard discounting.

(Laibson 1997; 1998; O'Donoghue and Rabin 1999); (ii) non-deterministic models with expected utility maximizing agents who violate Bayes' rule[4]; (iii) non-deterministic models with Choquet expected utility decision makers (Schmeidler 1989; Gilboa 1987) or/and Prospect theory decision makers (Tversky and Kahneman 1992; Wakker and Tversky 1993; Wakker 2010) who form conditional non-additive beliefs that may or may not be updated in accordance with some Bayesian update rule (Gilboa and Schmeidler 1993; Eichberger, Grant, and Kelsey 2007; 2012). Within the context of life-cycle models with time-discounting and survival uncertainty, our model with arbitrary effective discount factors contains these modeling classes as special cases.

How the strategic game of a sophisticated QHD agent against her future selves affects in a subgame-perfect Nash equilibrium the growth rate of marginal utility is reflected in the "generalized Euler equation" (Harris and Laibson 2001). The main difference between QHD preferences considered in Harris and Laibson (2001) and our model with arbitrary discount factors is that from the perspective of any model period h, the continuation problem at ages $h+2, h+3, \dots$ for current self h and future self $h+1$ for almost all specifications of the discounting process does not coincide with standard discounting (as long as period $h < T-2$). This implies that the "generalized Euler equation" must feature for non-QHD preferences an "adjustment factor" (Groneck, Ludwig and Zimper 2016), which, as a side result of our paper, we characterize analytically.

For the special case of the three-period model, our main result (Theorem 1) is already implied by the comparative statics analysis in Salanié and Treich (2006) (henceforth ST).[5] The three-period model allows for a very simple characterization of dynamic inconsistency through a single real-valued parameter. We show in Appendix B.1 that therefore the three period model can be analyzed by a first-order approach with respect to this parameter's deviation from the dynamic consistency benchmark case. The closely related comparative statics analysis in ST uses a single-crossing condition to look at one parameter deviations from the dynamic consistency benchmark case (cf. Appendix B.2). The problem with such first-order analysis—and with the, more general,

[4]The economic literature which considers violations of Bayesian updating includes Rabin and Schrag (1999); Rabin (2002); Epstein (2006); Epstein, Noor, and Sandroni (2008); Mullainathan, Schwartzstein, and Shleifer (2008); Gennaioli and Shleifer (2010); Ortoleva (2012). Bayesian updating of additive probability measures is, through the *law of iterated expectations*, equivalent to dynamic consistency of expected utility preferences (cf., e.g., Epstein and Le Breton 1993; Epstein and Schneider 2003; Ghirardato 2002; Siniscalchi 2011).

[5]We thank an anonymous referee for pointing this out. For the reader's convenience, we discuss in some detail the comparative statics analysis of Salanié and Treich (2006) in Appendix B.

comparative statics analysis—is that it becomes intractable for models with $T > 2$ because the number of parameters that are needed for capturing all possibilities of dynamic inconsistency blows up. In contrast, our recursive analysis avoids this limitation as we solve the problem for arbitrary $T < \infty$ from the start.

The extension of our main result to EZW preferences with uninsurable return risk is relevant for macroeconomic models such as the analytically tractable model with uninsurable investment risk developed by Angeletos (2007). Our findings also extend to a model with (additional) uninsurable human capital risk as developed in Krebs (2003). We further argue that with an according calibration of the rate of return process, this model gives rise to self-imposed borrowing constraint if there exists a strictly positive probability of loosing the wealth endowment from a risky investment. This extension may thus deliver a relevant quantitative environment with robust policy functions (Laibson and Maxted 2023)[6] to investigate the utility costs of dynamic inconsistency stemming from different specifications of discount functions in a model with uninsurable human capital income and return risk. We leave an investigation of this aspect for future research.

Maxted (2023) considers a continuous time, infinite time horizon model as introduced by Harris and Laibson (2013). Under the assumption of QHD preferences with a presence bias, this author describes (subgame-perfect) sophisticated decision making through a stationary Markov-equilibrium.[7] In contrast to our finite horizon model, for which we can establish through backward induction the existence of a unique subgame-perfect Nash equilibrium for arbitrary patterns of age-dependent discount factors, stationary Markov-equilibria require the stationarity of preferences in the specific sense that all future selves are simply replicas of an ex ante agent (as is the case for QHD preferences). But even restricted to QHD preferences with a presence bias, the infinite time horizon economy of Maxted (2023) is very different from our finite time-horizon economy. Namely, the sophisticated agent of Maxted (2023) saves in every period the same fraction of the naive agent's savings, which holds in our model only under dynamic consistency

[6]Laibson and Maxted (2023) show that in a standard buffer stock savings model consumption policy functions are "robust"—in a sense that they do not feature erratic non-monotonicities also labelled as "consumption pathologies"—if the time step with which a standard model is calibrated is less than about two weeks.

[7]When discount factor values become greater one, the contraction mapping theorem is no longer applicable to the effect that stationary Markov-equilibria with positive per-period consumption do not ncesseraily exist for QHD agents with a *future bias*.

or logarithmic utility (i.e., non-generic model specifications for which both agent types exhibit the same savings behavior). Given the substantial difference between the stationary Markov-equilibria in Maxted (2023) and the subgame-perfect Nash equilibria for our finite horizon model, it is remarkable that Maxted (2023) also obtains that his sophisticated agent saves at every point in time more than her naive counterpart if and only if $\theta \geq 1$.

Finally, our extension to standard income risk in the three-period model relates to Salanié and Treich (2006) who claim that Pollak (1968)'s observational equivalence result for the three-period model carries over to situations with income risk in the final period. We show that this claim is incorrect since this model extension implies a non-linear consumption policy function to the effect that the 0-old agent can strategically influence the choices of her 1-old future self.

The remainder of our analysis proceeds as follows. Section 2 solves the life-cycle model for the sophisticated and naive agent, respectively. To illustrate our main insights, we provide in Section 3 an in-depth analysis of the three-period model. Section 4 comprehensively answers our research question for the general T-period model and Section 5 discusses model extensions to environments with risk. Section 6 concludes. Mathematical proofs are relegated to Appendix A. Appendix B presents alternative proof ideas for our main result—based on first-order analysis and the comparative statics analysis in Salanié and Treich (2006), respectively—which are only tractable for the three-period model. Finally, Appendix C provides information on the calibration of our quantitative models.

2 The Life-Cycle Model

2.1 Baseline Model

We start out with an additively time-separable life-cycle model with final period $T < \infty$. For each age $h \geq 0$, the h-old agent's utility over the consumption stream $(c_h ..., c_T) \in \mathbb{R}^{T-h+1}_{>0}$ is given as

$$U_h (c_h, ..., c_T) = \sum_{t=h}^{T} \rho_{h,t} u (c_t) \tag{1}$$

such that the age-dependent effective discount factors must only satisfy $\rho_{h,t} > 0$ and $\rho_{h,h} = 1$. In our cake-eating baseline model the agent can spend over her life-cycle an initial amount of total wealth $w_0 > 0$ so that the budget constraint becomes[8]

$$w_{t+1} = w_t - c_t \geq 0 \quad \text{for } t \in \{0, 1, \ldots, T-1\}. \tag{2}$$

We restrict attention to period-utility functions belonging to the family of iso-elastic power utility functions, that is, $u(c)$ must be differentiable on $\mathbb{R}_{>0}$ such that

$$u(c) = \frac{c^{1-\theta}}{1-\theta}$$

and $u'(c) = c^{-\theta}$ for concavity parameter $0 < \theta < \infty$. For the special case of the three-period model, we will also investigate the limit cases $\theta \to 0$ (linear utility) and $\theta \to \infty$, respectively.

2.2 Sophisticated versus Naive Utility Maximization Problems

The existing literature on life-cycle models typically characterizes dynamically inconsistent discount factors first before it introduces sophisticated versus naive decision making as two different benchmark ways of how to deal with dynamic inconsistency. This approach is in particular the case for (i) quasi-hyperbolic time-discounting models, where dynamic inconsistency is characterized by the inequality $\beta \neq 1$ for the short-term discount factor, as well as for (ii) three-period models, where dynamic inconsistency is characterized by the inequality $\frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}} \neq 1$. Such simple characterizations of dynamic inconsistency are no longer available for our life-cycle model with arbitrary $T < \infty$ and arbitrary age-dependent discount factors. For our general model it is conceptually more rigorous to define at first sophisticated versus naive decision making. In a next step we use these definitions to offer a plausible definition of age-dependent dynamic consistency versus inconsistency.

2.2.1 Consumption Choices defined through a Best Response Function

For fixed period consumption c_t and wealth w_t let

$$c_t = m_t w_t \tag{3}$$

[8] In Section 5 we will consider several extensions of the baseline budget condition (2). Whereas our main findings will readily go through for budget extensions that preserve linearity of the consumption policy function (e.g., deterministic interest rates or random returns with portfolio choice), the situation will be more complex for budget extensions with income risk and borrowing constraints.

where m_t denotes the agent's *marginal propensity to consume* (MPC). Using the notational convention (3) for the periods $t, ..., T$ we can equivalently rewrite the utility (1) of the h-old agent from the consumption stream $(c_h, ..., c_T)$ as follows

$$U_h\left(m_h, ..., m_T, w_h\right) = u(w_h m_h) + \sum_{t=h+1}^{T} \rho_{h,t} u\left(w_h m_t \prod_{j=h}^{t-1}\left(1 - m_j\right)\right).$$

For the (homothetic) power-period utility function this utility representation is, by a monotone transformation, equivalent to

$$\hat{U}_h\left(m_h, ..., m_T\right) = \frac{U_h\left(m_h, ..., m_T, w_h\right)}{w_h^{1-\theta}} = u(m_h) + \sum_{t=h+1}^{T} \rho_{h,t} u\left(m_t \prod_{j=h}^{t-1}\left(1 - m_j\right)\right). \tag{4}$$

That is, the size of the wealth level w_h in period h does not matter for the utility maximization problem, which can exclusively be expressed in terms of MPCs. This observation corresponds to the well-known fact that the optimal period consumption is linear in total wealth for homothetic period utility functions.[9] The game-theoretic analysis of the extensive form game that corresponds to our baseline model is therefore particularly simple: because current optimal MPC choices are independent of past consumption choices, we can describe the utility maximization problem of the h-old agent through a best response function against anticipated future MPC choices only.[10]

Denote by $m_h^{*,h} : [0,1]^{T-h} \to [0,1]$ the h-old agent's *best response function* that maximizes the utility function (4) against anticipated future MPCs

$$\left(m_{h+1}, ..., m_T\right) \in [0,1]^{T-h}.$$

Proposition 1. *The MPC that is the best response of the h-old agent against the anticipated future choices $m_{h+1}, ..., m_T$ is given as the following value of her best response function:*

$$m_h^{*,h}\left(m_{h+1}, ..., m_T\right) = \frac{1}{1 + \left(\sum_{t=h+1}^{T} \rho_{h,t}\left(m_t \prod_{j=h+1}^{t-1}\left(1 - m_j\right)\right)^{1-\theta}\right)^{\frac{1}{\theta}}}. \tag{5}$$

[9] Linearity of consumption policy functions in models with a deterministic labor income stream and no borrowing constraints is a well-established result in the consumption literature, cf., e.g., Deaton (1992).

[10] Such 'path-independence' of best responses breaks down for model extensions with non-linear consumption policy functions (cf. Appendix A3 in Groneck, Ludwig and Zimper (2016) as well as Section 5.2 in this paper). In such model extensions an additional strategic dimension arises because an ex ante agent has now to consider the impact of her choices on the choices of her future selves.

In what follows we distinguish between an agent who is either sophisticated or naive throughout her whole life-cycle. The actual choice of an h-old agent of either type—i.e., their realized MPCs m_h^s and m_h^s, respectively—will be pinned down as their respective values of the best response function (5). Because both agents have identical preferences, they share the same best response function $m_h^{*,h}(\cdot)$. The difference between the two agent types is of cognitive or/and psychological nature: whereas the sophisticated agent uses correctly anticipated future choices as arguments in (5), her naive counterpart uses anticipated future choices that are optimal from her current perspective.

2.2.2 The Sophisticated Agent

Definition 1. *The "sophisticated agent" uses at every age h the actually realized MPCs of her future selves $\left(m_{h+1}^s, ..., m_T^s\right)$ as argument in the best response function (5); that is,*

$$m_h^s = m_h^{*,h}\left(m_{h+1}^s, ..., m_T^s\right). \tag{6}$$

We solve for the realized MPCs of the sophisticated agent through backward induction so that the life-cycle consumption choices of the sophisticated agent correspond to the subgame-perfect Nash equilibrium (SPNE) path of the underlying extensive form game. By Proposition 1, we obtain the following recursive characterization of the realized MPCs of the sophisticated agent.

Proposition 2. *The realized MPCs of the sophisticated agent are given as follows:*

$$m_h^s = \begin{cases} 1 & \text{for } h = T \\ \frac{1}{1+\left(\rho_{h,h+1}\zeta_{h+1}^h\right)^{\frac{1}{\theta}}} & \text{for } h \leq T-1 \end{cases} \tag{7}$$

where ζ_t^h scales the marginal valuation of wealth of self h in period t and is recursively defined as

$$\zeta_t^h = \begin{cases} 1 & \text{for } t = T \\ {m_t^s}^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}}\left(1 - m_t^s\right)^{1-\theta}\zeta_{t+1}^h & \text{for } t \leq T-1 \end{cases}$$

Interpretation in terms of Generalized Euler equations. To interpret the marginal propensities to consume against the literature on hyperbolic discounting, it is instructive to derive a *variant* of the *generalized Euler equation* (Harris and Laibson 2001) with an *adjustment factor* (Groneck, Ludwig and Zimper 2016), which as shown in Appendix A follows from the expressions for MPCs in Proposition 2 as

$$u_c(c_h^s) = \rho_{h,h+1} \left(m_{h+1}^s + \frac{\rho_{h,h+2}}{\rho_{h,h+1}\rho_{h+1,h+2}} \frac{\zeta_{h+2}^h}{\zeta_{h+2}^{h+1}} \left(1 - m_{h+1}^s\right) \right) u_c(c_{h+1}^s) \tag{8}$$

Equation (8) is the deterministic model analogue to the "generalized Euler equation with adjustment factor" we derived in a model with idiosyncratic productivity risk in Groneck, Ludwig and Zimper (2016). It reflects two effects on the consumption growth rate from dynamically inconsistent preferences. The first is through term

$$\frac{\rho_{h,h+2}}{\rho_{h,h+1}\rho_{h+1,h+2}} \neq 1 \text{ (in general)},$$

which in the familiar quasi-hyperbolic time discounting model is equal to $\frac{1}{\beta}$, where β is the short-run discount factor. The second is through the ratio

$$\frac{\zeta_{h+2}^h}{\zeta_{h+2}^{h+1}} \neq 1 \text{ (in general)},$$

which captures the difference in the marginal valuation of wealth in period $h+2$ from the perspective of the sophisticated agent h and her next period counterpart $h+1$. In the quasi-hyperbolic time discounting model $\frac{\zeta_{h+2}^h}{\zeta_{h+2}^{h+1}} = 1$, because continuation value functions from period $h+2$ onwards are the same for sophisticated agents h and $h+1$ in that model.

2.2.3 The Naive Agent

Denote by $\left(m_h^n, m_{h+1}^{n,h}, ..., m_T^{n,h}\right)$ the maximizer of the utility function (4). Obviously, for fixed 'planned' MPCs $\left(m_{h+1}^{n,h}, ..., m_T^{n,h}\right)$ the MPC m_h^n then maximizes (1) over all admissible MPCs at age h.

Definition 2. *The "naive agent" uses at every age h the most preferred future (i.e., planned) MPCs $\left(m_{h+1}^{n,h}, ..., m_T^{n,h}\right)$ as argument in the best response function (5); that is,*

$$m_h^n = m_h^{*,h}\left(m_{h+1}^{n,h}, ..., m_T^{n,h}\right). \tag{9}$$

The h-old sophisticated agent and her naive counterpart can only arrive at different solutions to their respective utility maximization problems if the most preferred (i.e., planned)

MPCs $\left(m_{h+1}^{n,h}, ..., m_{T}^{n,h}\right)$ do not coincide with the actually chosen MPCs $\left(m_{h+1}^{s}, ..., m_{T}^{s}\right)$ of the sophisticated agent.[11]

Proposition 3. *The realized MPCs of the naive agent are given as follows:*

(i) *Recursive characterization:*

$$m_h^n = \begin{cases} 1 & \text{for } h = T \\ \frac{1}{1+\left(\sum_{t=h+1}^{T} \rho_{h,t}\left(m_t^{n,h} \prod_{j=h+1}^{t-1}\left(1-m_j^{n,h}\right)\right)^{1-\theta}\right)^{\frac{1}{\theta}}} & \text{for } h \leq T-1 \end{cases}$$

with planned MPCs

$$m_t^{n,h} = \begin{cases} 1 & \text{for } t = T \\ \frac{1}{1+\frac{\rho_{t,t+1}^{\frac{1}{\theta}}}{m_{t+1}^{n,h}}} = \frac{1}{1+\sum_{k=t+1}^{T}\left(\frac{\rho_{h,k}}{\rho_{h,t}}\right)^{\frac{1}{\theta}}} & \text{for } t \leq T-1 \end{cases}$$

(ii) *Closed form:*

$$m_h^n = \frac{1}{1+\sum_{t=h+1}^{T}\left(\rho_{h,t}\right)^{\frac{1}{\theta}}} \text{ for } h \leq T-1.$$

2.2.4 Dynamic Consistency versus Inconsistency

We speak of dynamic inconsistency at age h whenever the h-old sophisticated agent and her naive counterpart anticipate different future choices. Formally, this is the case if and only if there exists any discrepancy between their respective arguments in their shared best response function $m_h^{*,h}(\cdot)$, i.e., if and only if

$$\left(m_{h+1}^{s}, ..., m_{T}^{s}\right) \neq \left(m_{h+1}^{n,h}, ..., m_{T}^{n,h}\right).$$

[11]The converse statement will only be true for $\theta \neq 1$ but not for $\theta = 1$ (cf. our Lemma 1). As a matter of fact, Pollak's (1968) famous behavioral equivalence result establishes for logarithmic utiltiy that the best response function $m_h^{*,h}(\cdot)$ gives identical values

$$m_h^{*,h}\left(m_{h+1}^{s}, ..., m_{T}^{s}\right) = m_h^{*,h}\left(m_{h+1}^{n,h}, ..., m_{T}^{n,h}\right)$$

in spite of $\left(m_{h+1}^{s}, ..., m_{T}^{s}\right) \neq \left(m_{h+1}^{n,h}, ..., m_{T}^{n,h}\right)$.

Definition 3.

(i) *The model is "dynamically consistent at age h" if*

$$m_t^s = m_t^{n,h} \text{ for all } t \geq h+1.$$

(ii) *Conversely, the model is "dynamically inconsistent at age h" if*

$$m_t^s \neq m_t^{n,h} \text{ for some } t \geq h+1.$$

The model is always dynamically consistent at ages $h \in \{T-1, T\}$. Because of $m_T^s = m_T^{n,T-2} = 1$, the model is dynamically consistent at age $h = T-2$ if and only if

$$\begin{aligned} m_{T-1}^s &= m_{T-1}^{n,T-2} \\ &\Leftrightarrow \\ \rho_{T-1,T} &= \frac{\rho_{T-2,T}}{\rho_{T-2,T-1}}. \end{aligned}$$

More generally, it is straightforward to show that the model is dynamically consistent at any given age $h \geq 0$ if the discount factors satisfy

$$\frac{\rho_{h,t+1}}{\rho_{h,t}} = \rho_{t,t+1} \text{ for all } t \in \{h+1, T-1\}. \tag{10}$$

For the QHD model condition (10) reduces to the familiar expression in terms of the short-term discount factor

$$\frac{\beta\delta^{t+1-h}}{\beta\delta^{t-h}} = \beta\delta \Leftrightarrow \beta = 1.$$

Whereas dynamic consistency in QHD and three-periods models is unambiguously characterized by (10), the situation is more complex for our more general set-up with arbitrary time periods and discount factors. First, condition (10) might no longer be necessary for dynamic consistency at age h. Second, alternative concepts of dynamic consistency become perceivable.[12]

At this point it is important to emphasize that our subsequent mathematical results about the savings behavior of naive versus sophisticated agents do not depend on any specific (formal

[12] In the earlier version of this paper (Groneck, Ludwig, and Zimper 2021) we had defined dynamic consistency as the coincidence between the realized and the anticipated consumption choices of the naive agent, without any reference to the anticipated consumption choices of the sophisticated agent. This alternative definition is, for more than three periods and non-QHD preferences, mathematically not equivalent to our present Definition 3.

or informal) definition of dynamic consistency versus inconsistency. These mathematical findings will be exclusively driven by our formal Definitions 1 and 2 of both agent types. We propose Definition 3 of dynamic consistency versus inconsistency merely for interpretational reasons and for better comparison with the existing literature.

3 The Simple Case of the Three-period Model

Our main result establishes for arbitrary $T < \infty$ that the effective discounting process is completely irrelevant for the question whether the naive or the sophisticated agent saves more in the presence of dynamic inconsistency. To make this remarkable result more easily accessible to readers, we here develop it in a three-period model ($T = 2$). In the three-period model dynamic inconsistency comes with a very simple structure. Because of

$$\begin{aligned} m_2^n &= m_2^s = 1, \\ m_1^n &= m_1^s = \frac{1}{1 + \left(\rho_{1,2}\right)^{\frac{1}{\theta}}}, \end{aligned}$$

it is sufficient to compare the respective MPCs for the initial period m_0^n and m_0^s whereby dynamic inconsistency is equivalently given by either of the following inequalities

$$m_1^{n,0} \neq m_1^n \quad \Leftrightarrow \quad m_1^{n,0} \neq m_1^s \quad \Leftrightarrow \quad \frac{\rho_{0,2}}{\rho_{0,1}} \neq \rho_{1,2}. \tag{11}$$

Our analysis of the three-period model is organized in four parts. The first part illustrates—applied to the simple three-period model—the recursive structure of the proof for the general T-period model analyzed in Section 4. The main insight of this formal argument is that the proof of our main result (Theorem 1) boils down to an application of Jensen's inequality. The second part illustrates how a shift from the dynamic benchmark case to dynamic inconsistency simultaneously changes the savings behavior of both agent types (thereby allowing for an inter-type and not just intra-type interpretation of this shift). The third part comprehensively discusses limit results for $\theta \to 0$ and $\theta \to \infty$. The main insight of this limit analysis is that the savings behavior of both agent types does not only converge if $\theta \to 1$ but also (i) if $\theta \to \infty$ and (ii) (under some additional restrictions on discount factors) if $\theta \to 0$. The fourth part illustrates our general findings through concrete examples in which the agents have quasi-hyperbolic time-discounting preferences whereby we look at different specifications of presence versus future biases.

3.1 Recursive Structure and Jensen's Inequality

The following formal argument—in terms of Jensen's inequality—carries over to the more complex proof by backward induction of our main result (Theorem 1). From our recursive characterization (7), we obtain for the three-period model

$$m_0^s = \frac{1}{1 + \left(\rho_{0,1}\zeta_1^0\right)^{\frac{1}{\theta}}}$$

with

$$\zeta_1^0 = m_1^{s^{1-\theta}} + \frac{\rho_{0,2}}{\rho_{0,1}} \left(1 - m_1^s\right)^{1-\theta} \tag{12}$$

as well as

$$m_0^n = \frac{1}{1 + \left(\frac{\rho_{0,1}}{\rho_{0,0}}\right)^{\frac{1}{\theta}} m_1^{n,0^{-1}}} \tag{13}$$

with

$$m_1^{n,0} = \frac{1}{1 + \left(\frac{\rho_{0,2}}{\rho_{0,1}}\right)^{\frac{1}{\theta}}}.$$

The sophisticated agent saves more than her naive counterpart in period 0 if and only if

$$\begin{aligned} m_0^n &\leq m_0^s \\ &\Leftrightarrow \\ \frac{1}{1 + \left(\frac{\rho_{0,1}}{\rho_{0,0}}\right)^{\frac{1}{\theta}} m_1^{n,0^{-1}}} &\leq \frac{1}{1 + \left(\rho_{0,1}\zeta_1^0\right)^{\frac{1}{\theta}}} \\ &\Leftrightarrow \\ \left(\rho_{0,1}\zeta_1^0\right)^{\frac{1}{\theta}} &\leq \left(\frac{\rho_{0,1}}{\rho_{0,0}}\right)^{\frac{1}{\theta}} m_1^{n,0^{-1}} \\ &\Leftrightarrow \\ m_1^{n,0^{\theta}} \zeta_1^0 &\leq 1. \end{aligned} \tag{14}$$

Rearranging (13) yields

$$\frac{\rho_{0,2}}{\rho_{0,1}} = \left(\frac{1 - m_1^{n,0}}{m_1^{n,0}}\right)^{\theta},$$

which, substituted in (12), gives

$$
\begin{aligned}
\zeta_1^0 &= m_1^{s^{1-\theta}} + \left(\frac{1-m_1^{n,0}}{m_1^{n,0}}\right)^\theta (1-m_1^s)^{1-\theta} \\
&\Leftrightarrow \\
m_1^{n,0^\theta}\zeta_1^0 &= m_1^{n,0^\theta} m_1^{s^{1-\theta}} + m_1^{n,0^\theta}\left(\frac{1-m_1^{n,0}}{m_1^{n,0}}\right)^\theta (1-m_1^s)^{1-\theta} \\
&\Leftrightarrow \\
m_1^{n,0^\theta}\zeta_1^0 &= \left(\frac{m_1^{n,0}}{m_1^s}\right)^\theta m_1^s + \left(\frac{1-m_1^{n,0}}{1-m_1^s}\right)^\theta (1-m_1^s)\,. \qquad (15)
\end{aligned}
$$

Combining (14) with (15) yields

$$
\begin{aligned}
m_0^n &\leq m_0^s \qquad (16) \\
&\Leftrightarrow \\
\left(\frac{m_1^{n,0}}{m_1^s}\right)^\theta m_1^s + \left(\frac{1-m_1^{n,0}}{1-m_1^s}\right)^\theta (1-m_1^s) &\leq 1. \qquad (17)
\end{aligned}
$$

Recall from (11) that our definition of dynamic inconsistency for the three-period model is equivalent to the inequality $m_1^{n,0} \neq m_1^s$. From Jensen's inequality we can therefore conclude that $m_0^n < (>)\, m_0^s$ if and only if $\theta < (>)\, 1$ whenever the three-period model is dynamically inconsistent.

To summarize, the discount process, which determines the values of $m_1^{n,0}, m_1^s \in (0,1)$, is completely irrelevant for the direction of the above inequality. The only thing that matters is whether the function $(\cdot)^\theta$ in (17) is either concave or convex, which is exclusively determined by the value of θ being smaller or greater than one.

3.2 Graphical Illustration and Interpretation

An interpretation of Pollak (1968)'s finding we often encountered in the literature is that of an interplay of the sophisticated agent's consumption and saving motives. On the one hand, in reaction to some dynamic inconsistency, e.g., a presence bias, the sophisticated consumer shifts resources more strongly to the current period in order to consume more in period 0. This is a standard inter-temporal substitution effect. On the other hand, the anticipated increasing consumption of her own future self reduces the purchasing power of overall resources in the future. This is similar to a standard income effect inducing the sophisticated agent to reduce her consumption in the current period to provide more resources to the future. This interpretation of Pollak's' finding, however, is incomplete because it only refers to the intra-personal decision of the sophisticated agent type but not to the inter-personal comparison between the decisions of the naive and sophisticated type. As it is, not only the sophisticated but also the naive agent reacts when the discount factors are shifted away from the dynamic consistency benchmark case.

To underscore this point, we here provide an alternative proof and a graphical interpretation of our finding which employs the Euler equations from both agents' maximization problems. From (8) the sophisticated agent's Euler equation is

$$(c_0^s)^{-\theta} = \rho_{0,1} \left(m_1 + (1-m_1) \frac{\rho_{02}}{\rho_{01}\rho_{12}} \right) (m_1 (w_0 - c_0^s))^{-\theta} \equiv RHS^s(c_0^s), \tag{18}$$

where

$$m_1 = m_1^s = m_1^n = \frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}. \tag{19}$$

For the naive household the standard Euler equation is

$$u_c(c_0^n) = \rho_{0,1} u_c\left(c_1^{0,n}\right). \tag{20}$$

Next, recall our solution of the MPC in period 1 for the naive agent:

$$m_1^{n,0} = \frac{1}{1+\left(\frac{\rho_{0,2}}{\rho_{0,1}}\right)^{\frac{1}{\theta}}}.$$

Using $\rho_{1,2}^{\frac{1}{\theta}} = \frac{1-m_1}{m_1}$ from (19) yields after some rearrangement for the above planned MPC

$$m_1^{n,0} = \frac{1}{m_1 + (1-m_1)\left(\frac{\rho_{02}}{\rho_{01}\rho_{12}}\right)^{\frac{1}{\theta}}} m_1. \tag{21}$$

Using (21) in Euler equation (20) yields

$$(c_0^n)^{-\theta} = \rho_{0,1} \left(m_1 + (1-m_1)\left(\frac{\rho_{02}}{\rho_{01}\rho_{12}}\right)^{\frac{1}{\theta}} \right)^{\theta} (m_1(w_0 - c_0^n))^{-\theta} \equiv RHS^n(c_0^n) \tag{22}$$

From (18) and (22) we immediately observe that for $\theta = 1$ the Euler equations coincide because the $RHS^s(\cdot)$ and $RHS^n(\cdot)$ are the same strictly increasing function in c_0

$$RHS(c_0) = \rho_{0,1} \left(m_1 + (1-m_1)\left(\frac{\rho_{02}}{\rho_{01}\rho_{12}}\right) \right) (m_1(w_0 - c_0))^{-1}$$

with $c_0^s = c_0^n$ being the unique fixed point of $RHS^{-1}(c_0)$. In contrast, for $\theta \neq 1$ we obtain instead that

$$c_0^s < c_0^n \qquad \Leftrightarrow \qquad m_1 + (1-m_1)\left(\frac{\rho_{02}}{\rho_{01}\rho_{12}}\right)^{\frac{1}{\theta}} > \left(m_1 + (1-m_1)\frac{\rho_{02}}{\rho_{01}\rho_{12}} \right)^{\frac{1}{\theta}}.$$

Our main result then again follows from a straightforward application of Jensen's inequality. This means that the substitution effect is relatively weaker for the sophisticated agent for $\theta > 1$, and relatively stronger for $\theta < 1$. We thus conclude that θ is an important parameter determining the relative shifts of the RHS in the first-order conditions of the two agent types and thus their relative consumption behavior in light of dynamic inconsistency.

For further interpretation, we parameterize the first-order conditions by assuming quasi-hyperbolic time discounting (QHD) such that for long-run discount factor $\delta \in (0, \infty)$, and short-run discount factor $\beta \in (0, \infty)$,

$$\rho_{0,1} = \beta\delta, \quad \rho_{1,2} = \rho_{0,1} = \beta\delta \quad \text{and} \quad \rho_{0,2} = \beta\delta^2 \quad \Leftrightarrow \quad \frac{\rho_{02}}{\rho_{0,1}\rho_{1,2}} = \frac{1}{\beta},$$

where a presence bias corresponds to $\beta < 1$ and a future bias to $\beta > 1$. The Euler equations in (18) and (22) then rewrite as

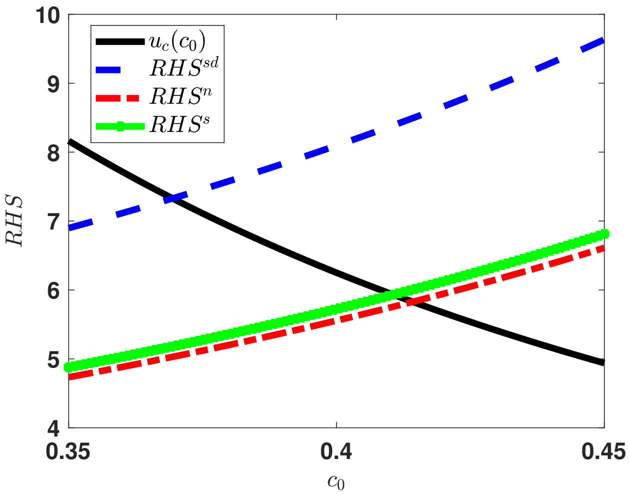
$$\begin{aligned} (c_0^s)^{-\theta} &= \delta\left(\beta m_1 + (1 - m_1)\right)\left(m_1\left(w_0 - c_0^s\right)\right)^{-\theta} \\ (c_0^n)^{-\theta} &= \delta\left(\beta^{\frac{1}{\theta}} m_1 + (1 - m_1)\right)^{\theta}\left(m_1(w_0 - c_0^n)\right)^{-\theta}, \end{aligned}$$

which, relative to a model that satisfies standard discounting at long-run discount factor δ, defines the proportional shift terms

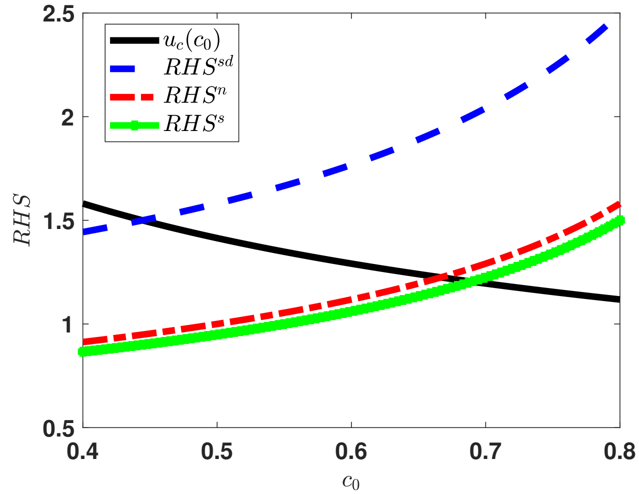
$$\begin{aligned} \Gamma^s &\equiv \beta m_1 + (1 - m_1) \\ \Gamma^n &\equiv \left(\beta^{\frac{1}{\theta}} m_1 + (1 - m_1)\right)^{\theta}. \end{aligned}$$

In a model with a presence bias ($\beta < 1$) both shift terms are smaller than one—so that, relative to standard discounting, both agents consume more in period 0—and in a model with a future bias ($\beta > 1$) both are larger than one—so that they both consume less than under standard discounting. Again by Jensen's inequality, we have, irrespective of the direction of the bias, $\Gamma^s > \Gamma^n$ for $\theta > 1$, whereas $\Gamma^s < \Gamma^n$ for $\theta < 1$. Figure 1 displays the LHS (black line), the RHS (blue line) of standard discounting, and the RHSs (red and green lines) of the Euler equations for the naive and the sophisticated agent for a presence bias for $\theta = 2$ in Panel (a) and for $\theta = 0.5$ in Panel (b), respectively. The corresponding first-order conditions for a future bias are shown in Panels (c) and (d).

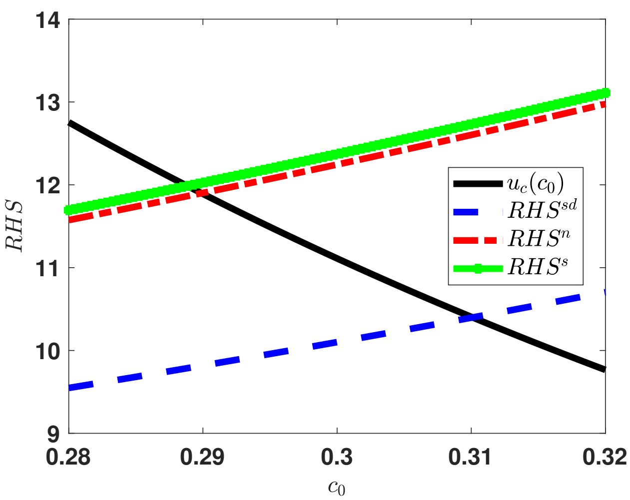
Figure 1: LHS and RHS's of the Euler Equation



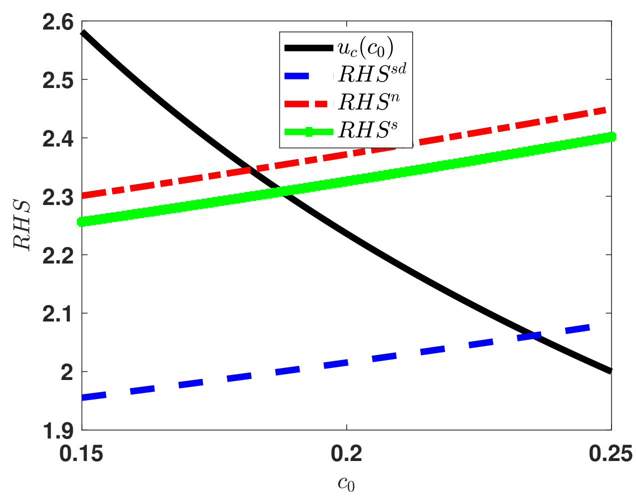
(a) Presence bias, $\theta = 2$



(b) Presence bias, $\theta = 0.5$



(c) Future bias, $\theta = 2$



(d) Future bias, $\theta = 0.5$

Notes: Period 0 Euler equation. $RHS^{sd}, RHS^{s}, RHS^{n}$: right-hand-side of Euler equation in model with *s*tandard *d*iscounting, as well as for the *s*ophisticated agent and the *n*ative agent, respectively. QHD model for parametrization $w_0 = 1$, $\delta = 1$, $\beta = 0.5$ in Panels (a) and (b), $\beta = 1.5$ in Panel (c) and (d), $\theta = 2$ in Panel (a) and (c), and $\theta = 0.5$ in Panel (b) and (d).

3.3 Limit Analysis

Since the central inequality (16) is exclusively driven by Jensen's inequality—with $\theta < (>)\,1$ determining a strictly concave (convex) function $(\cdot)^\theta$ in (17)—one might expect that any strictly positive difference $m_0^n - m_0^s > 0$ would be strictly increasing on the interval $(1, \infty)$. Analogously, one might also expect that smaller values of $\theta < 1$ result in an increase in the strictly positive difference $m_0^s - m_0^n > 0$. In what follows we show that this is (with the possible exception for the limit $\theta \to 0$) not the case.

Limit Analysis for $\theta \to \infty \quad \Leftrightarrow \quad \frac{1}{\theta} \to 0$. We establish, by the intermediate value theorem, that the maximal difference $m_0^n - m_0^s > 0$ must be obtained at some $\theta \in (1, \infty)$. To see this, rewrite the MPCs of both agent types (at age 0) as follows

$$
\begin{aligned}
m_0^n &= \frac{1}{1 + \rho_{0,1}^{\frac{1}{\theta}} + \rho_{0,2}^{\frac{1}{\theta}}}, \\
m_0^s &= \frac{1}{1 + \rho_{0,1}^{\frac{1}{\theta}} \left(\left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}} \right)^{1-\theta} + \rho_{1,2}^{\frac{1}{\theta}} \left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}} \right)^{1-\theta} \frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}} \right)^{\frac{1}{\theta}}}.
\end{aligned}
$$

Taking the limit yields for the naive agent $\lim_{\theta\to\infty} m_0^n = \frac{1}{3}$. For the sophisticated agent we have

$$
\begin{aligned}
\lim_{\theta\to\infty} \frac{1 - m_0^s}{m_0^s} &= \lim_{\theta\to\infty} \rho_{0,1}^{\frac{1}{\theta}} \left(\left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}} \right)^{1-\theta} + \rho_{1,2}^{\frac{1}{\theta}} \left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}} \right)^{1-\theta} \frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}} \right)^{\frac{1}{\theta}} \\
&= \rho_{0,1}^0 \left(\frac{1}{1+\rho_{1,2}^0} \right)^{-1} \left(1 + \rho_{1,2}^0 \frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}} \right)^0 = 2,
\end{aligned}
$$

which also yields $\lim_{\theta\to\infty} m_0^s = \frac{1}{3}$.

This shows that in the limit $\theta \to \infty$ consumption levels across periods become for both agent types perfect complements irrespective of the values of the discount factors as the IES $\frac{1}{\theta}$ converges towards zero. Consequently, the two types of agents exhibit in the limit identical consumption behavior in all periods, i.e., $c_h^s = c_h^n = \bar{c} = \frac{w_0}{3}$. While the strictly positive difference $m_0^n - m_0^s > 0$—which is a continuous function in $\theta \in (0, \infty)$—must thus strictly increase on some interval $[1, x]$ with $x > 1$ whenever the model is dynamically inconsistent, it will eventually decrease towards zero if θ gets large.

Limit Analysis for $\theta \to 0 \quad \Leftrightarrow \quad \frac{1}{\theta} \to \infty$. For the naive agent we obtain

$$\lim_{\theta\to 0} m_0^n = \lim_{\theta\to 0} \frac{1}{1+\rho_{0,1}^{\frac{1}{\theta}}+\rho_{0,2}^{\frac{1}{\theta}}} = \begin{cases} 1 & \text{if } \rho_{0,1}<1 \text{ and } \rho_{0,2}<1 \\ \frac{1}{3} & \text{if } \rho_{0,1}=1 \text{ and } \rho_{0,2}=1 \\ 0 & \text{if } \rho_{0,1}>1 \text{ or } \rho_{0,2}>1 \\ \frac{1}{2} & \begin{array}{c} \text{if } \rho_{0,1}=1 \text{ and } \rho_{0,2}<1, \text{ or} \\ \text{if } \rho_{0,1}<1 \text{ and } \rho_{0,2}=1. \end{array} \end{cases} \tag{23}$$

To analyze the situation for the sophisticated agent, observe that her utility function is continuous in period-0 consumption as well as in the concavity parameter $\theta \in [0,\infty)$ whereby we now include zero as a possible value. By Berge's (1997, p.116) maximum theorem, the maximizer correspondence $m_0^s : [0,\infty) \rightrightarrows [0,1]$ is upper-hemicontinuous in θ. Moreover, m_0^s is uniquely pinned down by the first-order condition for all $\theta \in (0,\infty)$ so that the upper-hemicontinuous correspondence $m_0^s(\theta)$ reduces to a continuous function on $(0,\infty)$. Generically, the correspondence $m_0^s(\theta)$ is also single-valued at $\theta = 0$, in which case upper-hemicontinuity implies[13]

$$\lim_{\theta\to 0} m_0^s(\theta) = m_0^s(0)$$

where $m_0^s(0)$ denotes the unique maximizer of the sophisticated agent's utility maximization problem at $\theta = 0$. Because of

$$u(c) = \frac{c^{1-0}}{1-0} = c,$$

this utility maximization problem at $\theta = 0$ is linear, which yields the following corner solutions (where we restrict attention to the generic case $\rho_{0,1}, \rho_{0,2}, \rho_{1,2} \neq 1$):

$$\begin{aligned}
\rho_{0,1} &> 1 \text{ and } \rho_{0,2} > 1 \text{ implies } m_0^s(0) = 0, \\
\rho_{0,1} &> 1 \text{ and } \rho_{0,2} < 1 \text{ combined with } \rho_{1,2} < 1 \text{ implies } m_0^s(0) = 0, \\
\rho_{0,1} &> 1 \text{ and } \rho_{0,2} < 1 \text{ combined with } \rho_{1,2} > 1 \text{ implies } m_0^s(0) = 1, \\
\rho_{0,1} &< 1 \text{ and } \rho_{0,2} < 1 \text{ implies } m_0^s(0) = 1, \\
\rho_{0,1} &< 1 \text{ and } \rho_{0,2} > 1 \text{ combined with } \rho_{1,2} < 1 \text{ implies } m_0^s(0) = 1, \\
\rho_{0,1} &< 1 \text{ and } \rho_{0,2} > 1 \text{ combined with } \rho_{1,2} > 1 \text{ implies } m_0^s(0) = 0.
\end{aligned}$$

[13] If $m_0^s(\theta)$ is multi-valued at $\theta = 0$, then upper-hemicontinuity implies for any non-zero sequence of θ

$$\lim_{\theta\to 0} m_0^s(\theta) = x \in m_0^s(0) = [0,1].$$

Multi-valuedness of $m_0^s(0)$ is non-generic as it happens iff (i) $\rho_{0,1} = 1$ and $\rho_{0,2} = 1$, or (ii) $\rho_{0,1} = 1$ and $\rho_{1,2} < 1$, or (iii) $\rho_{0,1} > 1$ and $\rho_{0,2} = 1$ combined with $\rho_{1,2} > 1$.

Unlike in the limit case $\theta \to \infty$, where

$$\lim_{\theta \to \infty} (m_0^s - m_0^n) = 0$$

holds for all possible discount factor values, the convergence behavior for $\theta \to 0$ can now be different for specific combinations of discount factor values. Combining the above conditions with (23) gives us the following three cases:

- **Limit Case 1.** We have

$$\lim_{\theta \to 0} (m_0^s - m_0^n) = 0 - 0 = 0$$

whenever

$$\begin{aligned} \text{(i) } \rho_{0,1} &> 1 \text{ and } \rho_{0,2} > 1, \text{ or} \\ \text{(ii) } \rho_{0,1} &> 1 \text{ and } \rho_{0,2} < 1 \text{ combined with } \rho_{1,2} < 1, \text{ or} \\ \text{(iii) } \rho_{0,1} &< 1 \text{ and } \rho_{0,2} > 1 \text{ combined with } \rho_{1,2} > 1. \end{aligned} \tag{24}$$

- **Limit Case 2.** We have

$$\lim_{\theta \to 0} (m_0^s - m_0^n) = 1 - 1 = 0$$

whenever

$$\text{(iv) } \rho_{0,1} < 1 \text{ and } \rho_{0,2} < 1. \tag{25}$$

- **Limit Case 3.** We have

$$\lim_{\theta \to 0} (m_0^s - m_0^n) = 1 - 0 = 1$$

whenever

$$\begin{aligned} \text{(v) } \rho_{0,1} &> 1 \text{ and } \rho_{0,2} < 1 \text{ combined with } \rho_{1,2} > 1, \text{ or} \\ \text{(vi) } \rho_{0,1} &< 1 \text{ and } \rho_{0,2} > 1 \text{ combined with } \rho_{1,2} < 1. \end{aligned} \tag{26}$$

For discount factor values covered by Cases 1 and 2, the two agent types exhibit in the limit $\theta \to 0$ identical consumption behavior in all periods, where the maximal difference $m_0^s - m_0^n > 0$ is obtained at some $\theta \in (0, 1)$. In contrast, for discount factor values covered by Case 3, the positive difference $m_0^s - m_0^n > 0$ converges on $(0, 1)$ to the maximal value of one if we let θ converge to zero.

3.4 Further Illustrative Examples

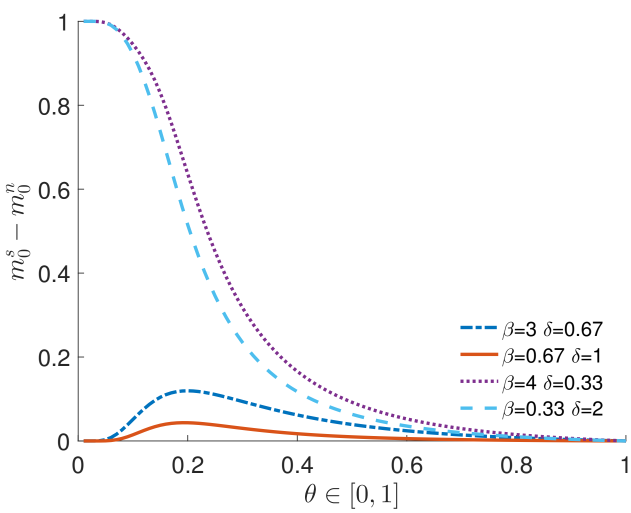
We again assume QHD such that for fixed parameters $\delta \in (0,\infty)$, $\beta \in (0,\infty)$, we get $\rho_{0,1} = \beta\delta, \rho_{1,2} = \rho_{0,1} = \beta\delta, \rho_{0,2} = \beta\delta^2$, where presence bias corresponds to $\beta < 1$ and future bias to $\beta > 1$. Recall that our main result implies that

$$\theta < (>)\, 1 \text{ implies } m_0^n < (>)\, m_0^s \text{ whenever } \frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}} = \frac{1}{\beta} \neq 1.$$

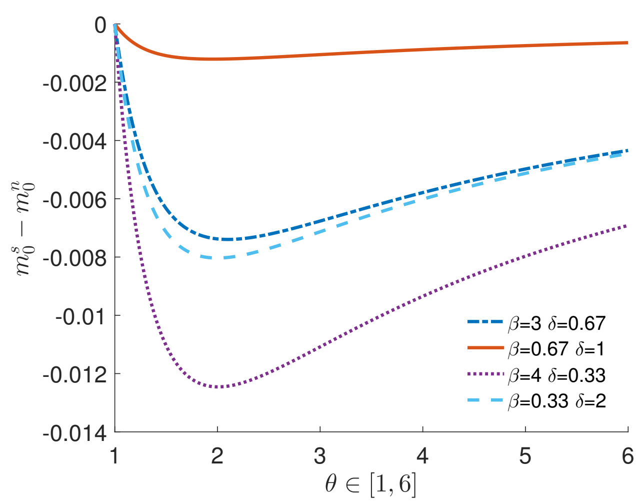
Limit Analysis for Quasi-hyperbolic Time-discounting. The previous limit analysis straightforwardly applies with the exception of Limit Cases 1(ii) and 1(iii) in equation (24), which are empty because of $\rho_{0,1} = \rho_{1,2} = \beta\delta$.

Illustration. Figure 2 displays the MPC differences $m_0^s - m_0^n$ for four parameterizations of the QHD model giving rise to the four possible Limit Cases 1(i), 2, 3(i) and 3(ii), respectively. To facilitate readability of the respective figure, we show the interval $\theta \in [0,1]$ in Panel (a) and the interval $\theta \in [1,6]$ in Panel (b).

Figure 2: Differences in MPCs in 3-Period Model



(a) $\theta \leq 1$



(b) $\theta \geq 1$

Notes: Percent differences between sophisticated and naive agents' MPCs for four parameterizations corresponding to the Limit Cases 1(i), 2, 3(i) and 3(ii) of Subsection 3.1, respectively.

4 Who Saves a Greater Fraction of Their Wealth?

4.1 Point of Departure

By definition, the sophisticated agent and her naive counterpart solve different utility maximization problems whenever the model is dynamically inconsistent. Quite surprisingly, however, the solutions to both problems coincide for arbitrary discount factors if the period-utility function is of the logarithmic form. This remarkable finding goes back to the seminal analysis in Pollak (1968).

Theorem 0 (Pollak 1968). *For all (arbitrary) specifications of the effective discount factors we have at every age h:*

$$\theta = 1 \textit{ implies } m_h^n = m_h^s.$$

It is straightforward to verify Pollak's Theorem directly by setting $\theta = 1$ in the MPCs of Propositions 2 and 3 to obtain

$$m_h^s = m_h^n = \begin{cases} 1 & \text{for } h = T \\ \frac{1}{1+\sum_{t=h+1}^{T} \rho_{h,t}} & \text{for } h \leq T-1. \end{cases}$$

4.2 New Results

For general $\theta \neq 1$ it follows also from Propositions 2 and 3 that the MPCs of the T- and $T-1$-old agents coincide for the naive and sophisticated type such that

$$\begin{aligned} m_T^n &= m_T^s = 1, \\ m_{T-1}^n &= m_{T-1}^s = \frac{1}{1+\left(\rho_{T-1,T}\right)^{\frac{1}{\theta}}}. \end{aligned}$$

For any ages $h \leq T-2$, however, it is no longer obvious how the sophisticated and naive agent's savings behavior will compare whenever $\theta \neq 1$. Our main result extends Pollak's analysis to the whole class of iso-elastic power utility functions, i.e., to all concavity parameter values $\theta \neq 1$.

Theorem 1. *For all (arbitrary) specifications of the effective discount factors we have at every age $h \leq T-2$:*

(i) $\theta < 1$ *implies* $m_h^n \leq m_h^s$;

(ii) $\theta > 1$ *implies* $m_h^n \geq m_h^s$.

The proof of Theorem 1 follows from the following Lemma, which is derived by a backward induction argument that uses Jensen's inequality.[14]

Lemma 1. *Let* $h \leq T-2$.

(i) $\theta < 1$ *implies* $m_h^n < m_h^s$ *if and only if* $m_t^{n,h} \neq m_t^s$ *for some* $t \geq h+1$.

(ii) $\theta > 1$ *implies* $m_h^n > m_h^s$ *if and only if* $m_t^{n,h} \neq m_t^s$ *for some* $t \geq h+1$.

(iii) $\theta \neq 1$ *and* $m_h^n = m_h^s$ *if and only if* $m_t^{n,h} = m_t^s$ *for all* $t \geq h+1$.

Game-theoretic Interpretation. Recall from our Definition 3 that $m_t^{n,h} \neq m_t^s$ for some $t \geq h+1$ means that the model is dynamically inconsistent at age h. That is, the h-old sophisticated agent and her naive counterpart solve different utility-maximization problems in the specific sense that their shared best response function (5) from Proposition 1 uses the different arguments $\left(m_{h+1}^s, ..., m_T^s\right)$ and $\left(m_{h+1}^{n,h}, ..., m_T^{n,h}\right)$, respectively. By Lemma 1, only the sophisticated—but not the naive—agent's savings behavior coincides with a subgame-perfect Nash equilibrium path if (i) the model is dynamically inconsistent and (ii) $\theta \neq 1$.

The situation is different for logarithmic utility. For $\theta = 1$ the best response function of the h-old agent (5) greatly simplifies to

$$m_h^{*,h}\left(m_{h+1}, ..., m_T\right) = \frac{1}{1 + \rho_{h,h+1} m_{h+1} + \cdots + \rho_{h,T} m_T}.$$

By Theorem 0 (Pollak 1968), it must therefore hold for $\theta = 1$ that

$$\begin{array}{rcl} m_h^s & = & m_h^n \\ & \Leftrightarrow & \\ m_h^{*,h}\left(m_{h+1}^s, ..., m_T^s\right) & = & m_h^{*,h}\left(m_{h+1}^{n,h}, ..., m_T^{n,h}\right) \\ & \Leftrightarrow & \\ \rho_{h,h+1} m_{h+1}^s + \cdots + \rho_{h,T-1} m_{T-1}^s & = & \rho_{h,h+1} m_{h+1}^{n,h} + \cdots + \rho_{h,T-1} m_{T-1}^{n,h} \end{array} \qquad (27)$$

for arbitrary discount factors. That is, the best responses against the different arguments $\left(m_{h+1}^s, ..., m_T^s\right)$ and $\left(m_{h+1}^{n,h}, ..., m_T^{n,h}\right)$ are always the same if $\theta = 1$. This leaves us with the

[14] For the basic proof idea see Section 3.1.

remarkable feature that for logarithmic utility the consumption choices of the naive agent are in hindsight mutually best responses, i.e., correspond to a (subgame-perfect) Nash equilibrium path, in spite of the fact that they were chosen as best responses against anticipated future choices that did not realize.

The 'precariousness' of the equality (27) suggests that it might break down when we move from our baseline model—in which best responses are 'path-independent'—to model extensions in which the h-old agent's choice has an impact on future choices. For such extensions we would therefore conjecture that the correct anticipation of this additional strategic impact by the sophisticated agent might make a difference in the solutions to both agents' utility maximization problems under logarithmic utility. We think that this is exactly what happens when we analyze in Section 5.2 the three-period model with period 2 income risk as considered in Salanié and Treich (2006). Due to this final period income risk, the 0-old agent has a strategic incentive to influence the 1-old's choice, whereby this impact will be correctly anticipated by the sophisticated but not by the naive agent. As it turns out, Theorem 0 (Pollak 1968) does no longer hold for this model extension. In contrast, we show that Theorem 0 (Pollak 1968) goes through for a model extension with income risk in period 1 only. In this situation, like in our baseline model, the 0-old agent cannot strategically influence the 1-old's MPC choice.

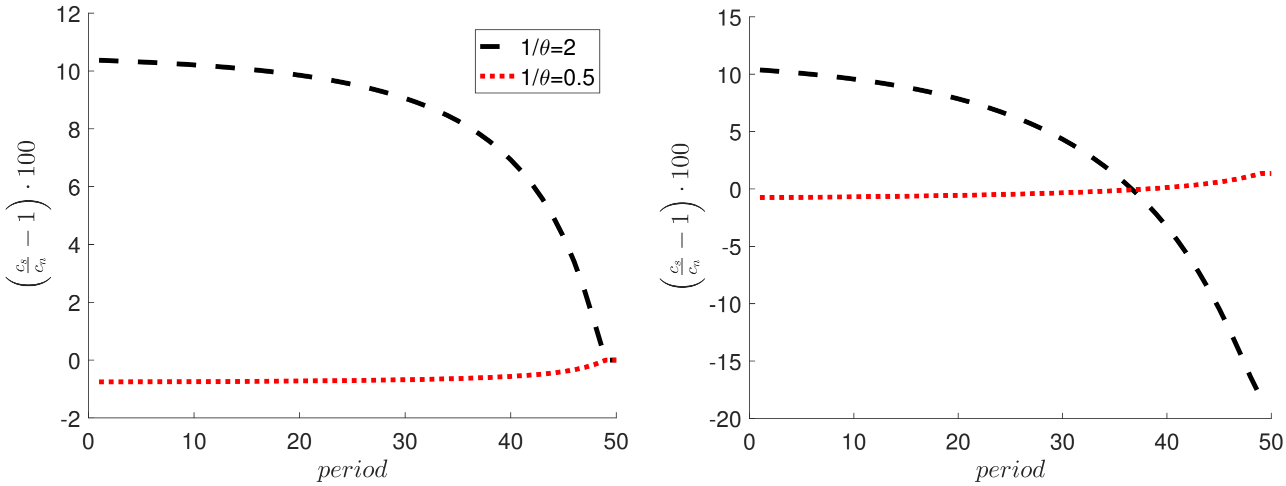
4.3 Quantitative Relevance

We illustrate the quantitative implications of our findings in Figure 3 for a stylized calibration of a standard hyperbolic time discounting model assuming estimates of θ regarded as plausible in the literature, cf. Bansal and Yaron (2004). We relegate to Appendix C a description of the calibration of the model. For a typical "finance" calibration with $1/\theta = 2$ the percent differences are very large, and also for a standard "macro" calibration where $1/\theta = 0.5$, the implied percent differences in consumption range from non-negligible -0.8% at age $h = 0$ to 1.4% at age T.

5 Discussion: Model Extensions

We first discuss model extensions that preserve linearity of the consumption policy function at all ages before we turn to model extensions which do not.

Figure 3: Percent Differences in Marginal Propensities to Consume and Consumption Levels



(a) Percent Differences in MPCs

(b) Percent Differences in Consumption

Notes: See Appendix C for the calibration of the model.

5.1 Epstein-Zin-Weil Preferences with Random Returns and Portfolio Choice

We extend our main analytical result to a life-cycle model with uninsurable return risk, a portfolio choice and Epstein-Zin-Weil (EZW) preferences (Epstein and Zin 1989; Epstein and Zin 1991; Weil 1989) with arbitrary discount factors. Our extension builds on fundamental insights of the seminal work by Merton (1969) and Samuelson (1969) who establish that the policy functions for consumption are linear in total wealth for homothetic preferences and serially uncorrelated returns. The according expressions for the MPCs of the naive and the sophisticated agent are therefore analogous to those in our baseline model. It is then straightforward to establish that the backward recursive proof of Theorem 1 readily extends to this setup.

5.1.1 Epstein-Zin-Weil Preferences

We express the familiar Epstein and Zin (1989, 1991) aggregator

$$V_t^h = \left(c_t^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \left(\mathbb{E}\left[V_{t+1}^{h^{1-\sigma}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \right)^{\frac{1}{1-\theta}}, \text{ for all } h \geq t$$

equivalently through the monotonic transformation

$$U_t^h = \frac{1}{1-\theta} V_t^{h^{1-\theta}} = u(c_t) + \frac{\rho_{h,t+1}}{\rho_{h,t}} \frac{1}{1-\theta} \left(\mathbb{E}\left[\left((1-\theta) U_{t+1}^h \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \tag{28}$$

for $u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$, where parameter $\sigma > 0$ is a measure of risk-aversion whereas parameter θ is a measure of resistance to inter-temporal substitution. For the parametrization $\sigma = \theta$ the standard additively time-separable case with CRRA per period utility function is nested as a special case.[15]

5.1.2 Random Return Process with Portfolio Choice

Let R_t be an independently (over time) distributed risky return factor governed by the additive probability measure π, where R_t takes weakly positive values π-almost surely. Additionally, let R^f be a risk-free return factor such that $R^f < \mathbb{E}[R_t] = \int R_t d\pi$. The household chooses in period $h \leq t$ to invest shares α_t in stocks with next period risky return R_{t+1} and $1 - \alpha_t$ in bonds with risk-free return R^f. The stochastic portfolio return on the beginning of period t financial wealth holdings is accordingly $R_t^p = R^f + \alpha_{t-1}\left(R_t - R^f\right)$. Additionally, let e_t be a (possibly time varying) deterministic endowment income stream of the agent.

The budget constraint in terms of financial wealth a_t is

$$a_{t+1} = a_t R_t^p(\alpha_{t-1}) + e_t - c_t \tag{29}$$

for a_0 given. In terms of cash on hand $x_t = a_t R_t^p(\alpha_{t-1}) + e_t$ we can rewrite the budget constraint as

$$x_{t+1} = (x_t - c_t)\, R_{t+1}^p(\alpha_t) + e_{t+1}. \tag{30}$$

Next, notice that human capital as the discounted sum of future deterministic labor income obeys

$$h_{t+1} = h_t R^f - e_{t+1}. \tag{31}$$

Consolidating budget constraints (30) and (31) gives

$$w_{t+1} = (w_t - c_t)\, R_{t+1}^p(\hat{\alpha}_t) \tag{32}$$

where

$$\hat{\alpha}_t = \alpha_t \frac{x_t - c_t}{w_t - c_t}. \tag{33}$$

[15] There exists an ongoing discussion in the literature regarding the question whether homothetic EZW preferences that explicitly incorporate the utility of possible death can be consistent with the natural assumption that 'life is better than death' for parameter values $\sigma \neq \theta$, $\sigma \geq 1, \theta \geq 1$ (cf. Hugonnier et al. 2013; Córdoba and Ripoll 2017; Bommier et al. 2020; Bommier et al. 2021). To sideline this discussion, we simply interpret the discount function $\rho_{h,t}$ as pure time discounting, thus $\rho_{h,t} = \delta_{h,t}$.

Note that if the event $R_{t+1} = 0$ has strictly positive probability, then by the lower Inada condition of the utility function, i.e. $\lim_{c\to 0} u_c(c) = \infty$, the household will choose $\hat{\alpha}_t < 1$ and thus there exists the possibility of self-imposed borrowing constraints. Also observe that a nested model variant is a cake-eating problem (i.e. $e_t = 0$ for all t) with risky returns (with or without a portfolio choice).

5.1.3 Solution

We derive in the Appendix A the following solutions to the EZW life-cycle model with random returns and portfolio choice for the naive and sophisticated agent, respectively.

Proposition 4. *Consider the EZW life-cycle model. The marginal propensities to consume are given as follows:*

- *for the sophisticated agent:*

$$m_h^{s,h} = \begin{cases} 1 & \text{for } h = T \\ \frac{1}{1+\left(\rho_{h,h+1}\zeta_{h+1}^h \Theta\left(\hat{\alpha}_t, R^f, R_{h+1}, \pi\right)\right)^{\frac{1}{\theta}}} & \text{for } h < T, \end{cases} \tag{34}$$

where ζ_{h+1}^h follows from the backward recursion in $t = T-1, \ldots, h$

$$\zeta_t^h = m_t^{s^{1-\theta}} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \left(1 - m_t^s\right)^{1-\theta} \zeta_{t+1}^h \cdot \Theta\left(\hat{\alpha}_t^*, R^f, R_{t+1}, \pi\right) \tag{35}$$

for $\zeta_T^h = 1$, where for all $t = h, \ldots, T-1$

$$\Theta\left(\hat{\alpha}_t, R^f, R_{t+1}, \pi\right) = \max_{\hat{\alpha}_t} \left\{ \left(\int R_{t+1}^p(\hat{\alpha}_t)^{1-\sigma} d\pi \right)^{\frac{1-\theta}{1-\sigma}} \right\}. \tag{36}$$

- *for the naive agent:*

$$m_t^{n,h} = \begin{cases} 1 & \text{for } t = T \\ \frac{1}{1+\left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\Theta\left(\hat{\alpha}_t, R^f, R_{t+1}, \pi\right)\right)^{\frac{1}{\theta}} \left(m_{t+1}^{n,h}\right)^{-1}} & \text{for } t < T, \end{cases} \tag{37}$$

where $\Theta(\cdot)$ is given by (36).

- *for both agents the optimal portfolio choice $\hat{\alpha}_t^s = \hat{\alpha}_t^n = \hat{\alpha}_t$ is the solution to*

$$\int R_{t+1}^p(\hat{\alpha}_t)^{-\sigma} d\pi = 0 \tag{38}$$

We thus find that the separation between risk attitudes as measured by σ and inter-temporal attitudes as measured by θ inherent to EZW preferences is reflected in the solution of this model to the effect that both households choose the same optimal portfolio share $\hat{\alpha}_t$ as the solution to (38)—which due to the convexity of the function $R^p_{t+1}(\hat{\alpha}_t)^{-\sigma}$ in the portfolio share is decreasing in risk aversion σ—, whereas the relationship between the marginal propensities to consume out of total wealth across the two types of households is exclusively driven by inter-temporal attitudes as measured by θ. Specifically, as in our recursive proof of Lemma 1 we likewise find that

$$m^n_h \leq m^s \qquad \Leftrightarrow \qquad \left(m^{n,h}_{h+1}\right)^{\theta} \zeta^h_{h+1} \leq 1$$

and since

$$\frac{\rho_{h,t+1}}{\rho_{h,t}} \Theta\left(\hat{\alpha}_t, R^f, R_{t+1}, \pi\right) = \left(\frac{1 - m^{n,h}_t}{m^{n,h}_t}\right)^{\theta} m^{n,h}_{t+1}$$

we can use the above in equation (35) to obtain (48). An application of the analogous steps as in the backward recursive proof of Theorem 1 gives us:

Corollary 1. *Lemma 1 and Theorem 1 extend to the dynamically inconsistent EZW life-cycle model with arbitrary discount factors.*

Our finding on marginal propensities to consume in Theorem 1 combined with the finding of equal (across the two types) optimal portfolio shares $\hat{\alpha}_t$ leads us to the next observation regarding the portfolio shares as a fraction of financial wealth α^i_t for $i \in \{n, s\}$. Recall from the definition of $\hat{\alpha}^i_t$ in (33) that

$$\alpha^i_t = \hat{\alpha}_t \left(1 + \frac{h_t}{x^i_t - c^i_t}\right)$$

and since (the optimal) $\hat{\alpha}_t$ and h_t are the same for both types of households, differences in the optimal portfolio choice out of financial wealth, α^i_t, across the two types are solely due to differences in $x^i_t - c^i_t$. Specifically, we get

$$\alpha^s_t \leq \alpha^n_t \qquad \Leftrightarrow \qquad x^s_t - c^s_t \geq x^n_t - c^n_t \qquad \Leftrightarrow \qquad w^s_t(1 - m^s_t) \geq w^n_t(1 - m^n_t).$$

Next, assume that the return realizations R_t are the same for the naive and the sophisticated household (aggregate return risk). Then, since at all t wealth accumulation, or decumulation, obeys (32) and since $\hat{\alpha}^i_t = \hat{\alpha}_t$, for $i \in \{n, s\}$ we obtain

$$m^s_t \leq m^n_t \;\Leftrightarrow\; (1 - m^s_t)w_t \geq (1 - m^n_t)w_t \;\Rightarrow\; w^s_{t+1} \geq w^n_{t+1},$$

for all $t \geq 0$, where the last inequality follows from (32) because $\hat{\alpha}_t^i = \hat{\alpha}_t$ and by our assumption of aggregate return risk so that return realizations are the same for the naive and the sophisticated household. This yields:

Corollary 2. *Theorem 1 extends to portfolio shares in the dynamically inconsistent EZW life-cycle model with arbitrary discount factors such that:*

(i) $\theta < 1$ *implies* $\alpha_h^n \leq \alpha_h^s$ *for all* h*;*

(ii) $\theta > 1$ *implies* $\alpha_h^n \geq \alpha_h^s$ *for all* h*.*

5.1.4 A Model with Risky Human Capital

An alternative model giving rise to the same mathematical properties as the above model with portfolio choice is one with risky labor income generated by risky returns to human capital h_t and a linear human capital production function taking monetary human capital investments i_t as inputs, cf. Krebs (2003). The dynamic budget constraint for asset accumulation is now given as

$$a_{t+1} = a_t R_t^p(\alpha_{t-1}) + r^h h_t - c_t - i_t \tag{39}$$

where r^h is some return on human capital h_t and i_t are monetary investments in human capital accumulation. Here, we take r^h as a deterministic number. Extensions to stochastic return factor r^h are, of course, possible. The human capital accumulation formula is assumed linear

$$h_{t+1} = h_t \left(1 + \zeta - \gamma\right) + i_t, \tag{40}$$

where ζ is an i.i.d. human capital accumulation shock and γ is the depreciation rate of human capital, both of which realize before the investment decision i_t is made. We provide more details on the assumed support of ζ below. Combining (39) and (40) gives, after some transformations of the budget constraint,

$$a_{t+1} + h_{t+1} = R_t^p\left(\hat{\alpha}_{t-1}, \hat{\alpha}_{t-1}^h\right)\left(a_t + h_t\right) - c_t,$$

where $\hat{\alpha}_{t-1} = \frac{\alpha_{t-1} a_t}{a_t + h_t}$ and $\hat{\alpha}_{t-1}^h = \frac{h_t}{a_t + h_t}$, $R_t^h(\zeta) = 1 + r^h + \zeta - \gamma$ is the stochastic return on human capital, and $R_t^p\left(\hat{\alpha}_{t-1}, \hat{\alpha}_{t-1}^h\right) = R^f + \hat{\alpha}_{t-1}\left(R_t - R^f\right) + \hat{\alpha}_{t-1}^h\left(R_t^h(\zeta) - R^f\right)$ is the portfolio return from holding all three assets (the risk-free asset, the risky asset and the risky human capital). We assume that $R_t^h(\zeta)$ is weakly positive ζ-almost-surely, i.e. $\zeta \geq \gamma - (1 + r^h)$.

Next, define by $w_t = R_t^p\left(\hat{\alpha}_{t-1}, \hat{\alpha}_{t-1}^h\right)(a_t + h_t)$ total wealth cum interest so that the budget constraint becomes

$$w_{t+1} = (w_t - c_t)\, R_{t+1}^p\left(\hat{\alpha}_t, \hat{\alpha}_t^h\right),$$

which is of the same form as (32). Thus, the separation result between the optimization over portfolio shares $\hat{\alpha}_t, \hat{\alpha}_t^h$—which in each $t < T$ is now a bivariate problem—and consumption c_t as above applies and consumption policy functions are linear functions in w_t. Finally, notice that if the event $R_{t+1} = R_{t+1}^h = 0$ has strictly positive probability in all periods, then by the lower Inada condition of the utility function, i.e. $\lim_{c\to 0} u_c(c) = \infty$, the household invests in all periods a strictly positive amount into the risk-free, thus $1 - \hat{\alpha}_t - \hat{\alpha}_t^h > 0$ (self imposed borrowing constraint).

5.2 Standard Labor Income Risk and Borrowing Constraints

The analysis of the previous subsection has established that our main result (Theorem 1) carries over to model extensions which preserve linearity of the consumption policy function. Linearity of consumption policy functions, however, does in general no longer hold for model extensions with standard (additive) income risk and borrowing constraints. In such models the consumption policy functions are concave for naive agents and for standard exponential discounting agents (Carroll and Kimball 1996). For sophisticated agents only continuity can be established (Harris and Laibson 2001) whereby it is well known that consumption policy functions may exhibit rather strong non-monotonicities (cf. Laibson and Maxted (2023) and the literature cited therein).

5.2.1 Preliminaries

To focus thoughts, we assume that the agent earns a stochastic income $y\eta$ that obeys a Markov process with transition matrix $\pi(\eta_{t+1} \mid \eta_t)$ and age 0 distribution $\Pi(\eta_0)$. The shock process of η_t is normalized such that mean income is y. Further, we impose an age-independent borrowing limit of the form $a_{t+1} \geq -\bar{A}$ for some $\bar{A} \in (-\infty, 0]$. Of course, the natural borrowing limit also applies. To preserve direct comparability to the deterministic cake eating problem of Section 4, we assume a zero interest rate.

The dynamic programming problem of an h-old sophisticated agent with value function $V_t^h(\cdot)$ and period t asset holdings a_t and income shock realization η_t is

$$V_h^h(a_h, \eta_h) = \max_{c_h^s, a_{h+1}} \left\{ u(c_h^s) + \rho_{h,h+1} \mathbb{E}_{\eta_{h+1}|\eta_h} \left[V_{h+1}^h \left(a_{h+1}, \eta_{h+1} \right) \right] \right\}$$

$$V_t^h(a_t, \eta_t) = \max_{c_t^s, a_{t+1}} \left\{ u(c_t^s) + \frac{\rho_{h,t+1}}{\rho_{h,h+1}} \mathbb{E}_{\eta_{t+1}|\eta_t} \left[V_{t+1}^h \left(a_{t+1}, \eta_{t+1} \right) \right] \right\}, \qquad \text{for all } t = h+1, \ldots, T$$

with terminal condition normalized to $V_{T+1}^h(a_{T+1}, \eta_{T+1}) = 0$. Maximization is subject to

$$V_{h+1}^{h+1}(a_{h+1}, \eta_{h+1}) = \max_{c_{h+1}^s, a_{h+2}} \left\{ u(c_{h+1}^s) + \rho_{h+1,h+2} \mathbb{E}_{\eta_{h+2}|\eta_{h+1}} \left[V_{h+2}^{h+1} \left(a_{h+2}, \eta_{h+2} \right) \right] \right\} \tag{41a}$$

$$a_{t+1} = a_t + y_t - c_t^s, \text{ where } y_t = \begin{cases} y\eta & \text{for } t < t_r \\ b & \text{otherwise.} \end{cases} \quad \text{for all } t = h, \ldots, T$$

$$a_{t+1} \geq -\bar{A}, \tag{41b}$$

$$a_{T+1} \geq 0, \tag{41c}$$

whereby due to no satiation the transversality condition (41c) holds with equality in the optimum. Constraint (41a) reflects that the sophisticated agent anticipates her future self's deviating preferences. The naive agent solves a similar problem but does not anticipate the deviation of her own future self, thus constraint (41a) is absent for the naive agent.

5.2.2 Three-Period Model with Logarithmic utility

As in the analysis of the deterministic model, we first consider a simple three-period variant. To this purpose, WOLG, we normalize $y = \frac{1}{3}$ so that the deterministic human wealth in the model, $w_0 = 3y$, is normalized to $w_0 = 1$. We further set $a_0 = 0$ and $\eta_0 = 1$, i.e. households are born with zero financial assets and there is no ex-ante heterogeneity. In the sequel we consider different parameterizations of this model. We start by assuming $\bar{A} = \infty$ and $\eta_2 = 1$ so that there are no borrowing constraints and income is only risky in period 1. We then consider the case $\bar{A} = \infty$ and $\eta_1 = 1$ so that there are no borrowing constraints and income is only risky in period 2.

In what follows we focus on logarithmic per-period utility and show that Pollak (1968)'s observational equivalence result (Theorem 0) breaks down for model extensions in which labor income risk implies that ex ante agents can strategically influence the choices of their future selves.

Income Risk in Period 1. Setting $\eta_2 = 1, \bar{A} = \infty$ the model is formally equivalent to a model with random wealth whose true value is revealed after the MPC decision in period 1. Denote by $w_0 = 3y$ the deterministic wealth component in the first period of life, so that in the second period the resources available for consumption are $w_1 = (1 - m_0^i)w_0 - y + y\eta_1$. Next introduce random variable $\epsilon_1 = y(\eta_1 - 1)$ so that $y\eta_1 = y + \epsilon_1$ to rewrite $w_1 = (1 - m_0^i)w_0 + \epsilon_1$. Since there is no income risk in the last period of life, the optimal consumption policy function in period 1 is, for both agent types

$$c_1^i = m_1 \left((1 - m_0^i)w_0 + \epsilon_1 \right), \text{ where } m_1 = \frac{1}{1 + \rho_{1,2}^{\frac{1}{\theta}}}$$

for $i \in \{n, s\}$. As in our baseline model, the consumption policy function at age 1 is linear in wealth $w_1 = ((1 - m_0^i)w_0 + \epsilon_1)$ to the effect that the 0-old agent cannot strategically influence the MPC choice of her 1-old future self.

Simply extending our analysis in Subsection 3.2 by the expectations operator, yields for logarithmic utility the following Euler equation for the sophisticated agent

$$\frac{1}{m_0^s w_0} = \rho_{0,1} \left(m_1 + (1 - m_1) \left(\frac{\rho_{02}}{\rho_{01}\rho_{12}} \right) \right) \mathbb{E} \left[\frac{1}{m_1 \left((1 - m_0^s)\, w_0 \right) + \epsilon_1} \right]$$

and for the naive agent

$$\frac{1}{m_0^n w_0} = \rho_{0,1} \left(m_1 + (1 - m_1) \left(\frac{\rho_{02}}{\rho_{01}\rho_{12}} \right) \right) \mathbb{E} \left[\frac{1}{m_1 \left((1 - m_0^n)\, w_0 \right) + \epsilon_1} \right].$$

Note that the consumption policy function at age 0 is no longer linear in w_0. However, as there is no agent before the 0-old agent there is no agent who could strategically impact on the 0-old agent's choice. As a consequence, the analysis our baseline model goes through. To be specific, by the same fixed point argument as in Subsection 3.2, we have $m_0^s = m_0^n$ because m_0^s and m_0^n are both the unique fixed point of the strictly increasing function $RHS^{-1}(\cdot)$ in m_0 such that

$$RHS(m_0) = \rho_{0,1} \left(m_1 + (1 - m_1) \left(\frac{\rho_{02}}{\rho_{01}\rho_{12}} \right) \right) \mathbb{E} \left[\frac{1}{m_1 \left((1 - m_0)\, w_0 \right) + \epsilon_1} \right] w_0.$$

As in our baseline model, $\theta = 1$ thus also remains the watershed case for the three-period model with income risk in period 1 because no agent can strategically influence her future selves.

Income Risk in Period 2. Setting $\eta_1 = 1, \bar{A} = \infty$ the model is formally equivalent to a model with random wealth whose true value is revealed after the MPC decision in period 2, which is already discussed in Salanié and Treich (2006) (henceforth ST). To make our own analysis better comparable to ST, we are going to use their formal framework. Let $y\eta_2 = y + \epsilon_2$, i.e., let the random wealth shock be $\epsilon_2 = y(\eta_2 - 1)$, so that total wealth $W_0 = w_0 + \epsilon_2$ is a random variable whose true value will only be revealed in the final period. The corresponding utility functions in ST for the 0-old and 1-old agents become

$$\begin{aligned} U_0(c_0, c_1) &= u(c_0) + \rho_{01} u(c_1) + \rho_{02} \mathbb{E}\left[u(W_0 - c_1 - c_0)\right], \\ U_1(c_0, c_1) &= u(c_1) + \rho_{12} \mathbb{E}\left[u(W_0 - c_1 - c_0)\right]. \end{aligned}$$

ST write (cf. p.1564): "Finally, an interesting implication of the condition on the utility function derived in Proposition 1 is that it makes immediate the generalization of the comparative statics to conditions of uncertainty. ... As a result, our condition is left unchanged when there is some uncertainty on future revenues." ST thus claim that under wealth uncertainty, just as in the deterministic case, the naive and sophisticated agent consume the same amount as long as they have a logarithmic period-utility function. In what follows we show that this claim is false.[16]

Let $u(c) = \ln(c)$ and suppose that the wealth shock is given as

$$\epsilon_2 = \begin{cases} -\nu & \text{with prob. } \frac{1}{2} \\ +\nu & \text{with prob. } \frac{1}{2} \end{cases}$$

for some $\nu \in (0,1)$. For a fixed c_0 the optimal period 1 consumption of the 1-old agent of both agent types is then pinned down by the FOC

$$c_1^* = \frac{1}{\rho_{12}\mathbb{E}\left(\frac{1}{W_0 - c_0 - c_1^*}\right)} = \frac{1}{\rho_{12}\frac{1}{2}\left[\frac{1}{\ln((1-\nu)-c_1^*-c_0} + \frac{1}{\ln((1+\nu)-c_1^*-c_0}\right]}$$

with solution

$$c_1^*(c_0) = \left(\frac{(2+\rho_{12})}{2(1+\rho_{12})}(1-c_0)\right) - \sqrt{\left(\frac{(2+\rho_{12})}{2(1+\rho_{12})}(1-c_0)\right)^2 - \frac{(1-c_0)^2 - \nu^2}{(1+\rho_{12})}}. \tag{42}$$

Because the 0-old sophisticated agent correctly anticipates her future consumption behavior in dependence on c_0, she chooses c_0^s as the maximizer of the following utility function

$$\begin{aligned} U_0(c_0, c_1^s(c_0)) &= \ln c_0 + \rho_{01}\ln(c_1^s(c_0)) \\ &\quad + \rho_{02}\frac{1}{2}\left[\ln((1-\nu) - c_1^s(c_0) - c_0) + \ln((1+\nu) - c_1^s(c_0) - c_0)\right] \end{aligned} \tag{43}$$

[16] To support their claim, ST provide an informal argument which is incomplete. Instead of considering both value functions for the periods 1 and 2, ST only consider the final period 2.

such that $c_1^s(c_0) = c_1^*(c_0)$. In contrast, the naive agent chooses c_0^n as the maximizer of

$$\begin{aligned} U_0\left(c_0, c_1^{n,0}(c_0)\right) &= \ln c_0 + \rho_{01}\ln\left(c_1^{n,0}(c_0)\right) \\ &+\rho_{02}\frac{1}{2}\left[\ln((1-\nu) - c_1^{n,0}(c_0) - c_0) + \ln((1+\nu) - c_1^{n,0}(c_0) - c_0)\right] \end{aligned} \tag{44}$$

whereby the naive's planned consumption for period 1 is given as

$$\begin{aligned} c_1^{n,0} &= \frac{1}{\frac{\rho_{02}}{\rho_{01}}\mathbb{E}\left(\frac{1}{W_0-c_0-c_1^{n,0}}\right)} \\ &\Rightarrow \\ c_1^{n,0}(c_0) &= \left(\frac{\left(2+\frac{\rho_{02}}{\rho_{01}}\right)}{2\left(1+\frac{\rho_{02}}{\rho_{01}}\right)}(1-c_0)\right) - \sqrt{\left(\frac{\left(2+\frac{\rho_{02}}{\rho_{01}}\right)}{2\left(1+\frac{\rho_{02}}{\rho_{01}}\right)}(1-c_0)\right)^2 - \frac{(1-c_0)^2-\nu^2}{\left(1+\frac{\rho_{02}}{\rho_{01}}\right)}}. \end{aligned} \tag{45}$$

For the deterministic baseline case $\nu = 0$ we have linearity of the consumption policy functions in $w_1 = 1 - c_0$ so that (42) and (45), respectively, reduce to our familiar results in terms of MPCs

$$\begin{aligned} c_1^s(c_0) &= \frac{1}{(1+\rho_{12})}(1-c_0) \Leftrightarrow m_1^s = \frac{1}{(1+\rho_{12})} \text{ and} \\ c_1^{n,0}(c_0) &= \frac{1}{\left(1+\frac{\rho_{02}}{\rho_{01}}\right)}(1-c_0) \Leftrightarrow m_1^{n,0} = \frac{1}{\left(1+\frac{\rho_{02}}{\rho_{01}}\right)}. \end{aligned}$$

By our previous analysis, this implies $c_0^s = c_0^n$ for the deterministic base-line model with $\nu = 0$ and logarithmic utility. For $\nu > 0$, however, this linearity breaks down because these MPCs become non-trivial functions in current wealth $w_1 = \mathbb{E}(W_0) - c_0 = 1 - c_0$. More relevantly to our game-theoretic interpretation, due to $w_1 = 1 - m_0 w_0$ these 1-old MPCs become for fixed wealth w_0 functions in the 0-old's MPC choice m_0 whenever $\nu > 0$:

$$\begin{aligned} m_1^s(m_0) &= \frac{c_1^s(c_0)}{w_1} = \frac{(2+\rho_{12})}{2(1+\rho_{12})} - \sqrt{\left(\frac{(2+\rho_{12})}{2(1+\rho_{12})}\right)^2 - \frac{1-\frac{\nu^2}{(1-m_0w_0)^2}}{(1+\rho_{12})}}, \\ m_1^{n,0}(m_0) &= \frac{c_1^{n,0}(c_0)}{w_1} = \frac{\left(2+\frac{\rho_{02}}{\rho_{01}}\right)}{2\left(1+\frac{\rho_{02}}{\rho_{01}}\right)} - \sqrt{\left(\frac{\left(2+\frac{\rho_{02}}{\rho_{01}}\right)}{2\left(1+\frac{\rho_{02}}{\rho_{01}}\right)}\right)^2 - \frac{1-\frac{\nu^2}{(1-m_0w_0)^2}}{\left(1+\frac{\rho_{02}}{\rho_{01}}\right)}}. \end{aligned}$$

This additional strategic complexity is fully understood by the sophisticated agent whose 0-old self correctly anticipates through the function value $m_1^s(m_0)$ the impact of her choice on the choice of her 1-old future self. In contrast, the naive agent does not understand this additional strategic complexity as her 0-old self remains completely ignorant about the fact that she is involved in a strategic game with her 1-old future self. The resulting differences between $m_1^s(m_0)$ and $m_1^{n,0}(m_0)$ for all different values of m_0 and all $\nu > 0$ are a measure for this ignorance. Due to this additional strategic complexity—as compared to our baseline model with $v = 0$—we would conjecture that ST's claim that $c_0^s = c_0^n$ holds for arbitrary $\nu > 0$ cannot be correct. And indeed, a simple quantitative exercise which directly calculates the maximizers of (43) and (44), respectively, shows that the naive and the sophisticated agent do not consume the same. To this purpose, we parameterize the model, again assuming the familiar quasi-hyperbolic time discounting setup. Specifically, we choose for the long-run discount factor $\delta = 1$ and for the short-run discount factor $\beta = 0.5$ to set a presence bias, respectively $\beta = 1.5$ to set a future bias. For the shock we assume $\nu = 0.8 \cdot y = 0.8 \cdot \frac{1}{3}$.

Table 1 provides the results where we focus attention only on period 0 consumption of the two agent types and display in the upper part of the table the results with a presence bias, respectively in the lower part those for a future bias. Column "NR" shows the results in the deterministic economy (No Risk), column "IR 1" those for income risk in the first period of life, and column "IR 2" those for the ST model with the wealth shock in period 2 (i.e., Income Risk only in period 2). In this latter, scenario, both with a presence as well as a future bias the sophisticated agent turns out to save more. While the differences are small, these differences are not a consequence of numerical round-off because the solution of the maximization problem of the sophisticated agent is computed at a much higher accuracy.[17]

General Income Risk and Borrowing Constraints. We extend the analysis by allowing for (i) income risk in the second period in column "IR 1+2" of Table 1, (ii) additionally a zero borrowing constraint ($\bar{A} = 0$) in column "IR 1+2, BC" and (iii) by autocorrelated risk as is typically assumed in quantitative work in column "AIR 1+2, BC". Note that the borrowing constraint is neither binding in period 0 nor in period 1, but leads to a reduction of period 0 consumption because of standard institutionally motivated precautionary savings: to smooth their consumption households save more in the presence of liquidity constraints to avoid future constraints to become binding. Also notice that the borrowing constraint is irrelevant with a future bias because of the strong inter-temporal (or life-cycle) savings motive. Throughout,

[17] We set the error tolerance to $1e-08$.

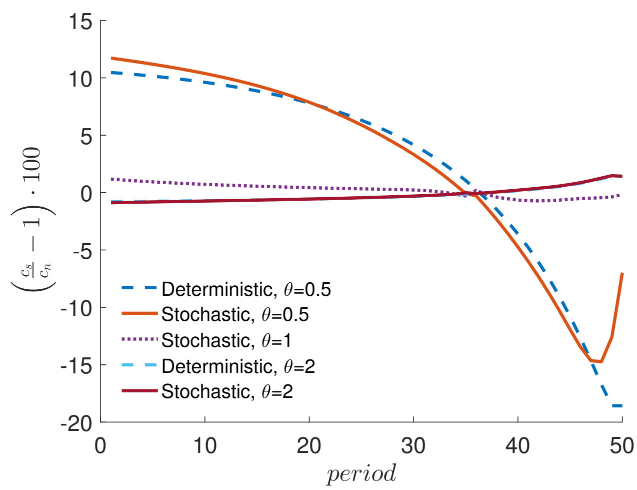
Table 1: Difference in Consumption Behavior $c_0^s - c_0^n$

Specification	NR	IR 1	IR 2	IR 1+2	IR 1+2, BC	AIR 1+2, BC
			Presence Bias, $\beta = 0.5$			
c_0^n	0.5	0.43686	0.40827	0.31564	0.30616	0.29979
c_0^s	0.5	0.43686	0.40965	0.31625	0.3063	0.31111
$c_0^s - c_0^n$	0.0	0.0	0.0013	0.0006	0.0001	0.011318
			Future Bias, $\beta = 1.5$			
c_0^n	0.25	0.227	0.21215	0.17714	0.17714	0.16607
c_0^s	0.25	0.227	0.21253	0.17742	0.17742	0.16622
$c_0^s - c_0^n$	0.0	0.0	0.00038	0.00027	0.00027	0.00015

Notes: NR: no risk, IR 1: income risk in period 1, IR 2: income risk in period 2 (ST model), IR 1+2: i.i.d. income risk in periods 1 and 2, IR 1+2, BC: i.i.d. income risk in periods 1 and 2 plus zero-borrowing constraint, AIR 1+2, BC: autocorrelated income risk in periods 1 and 2 plus zero-borrowing constraint.

we find that with logarithmic utility the sophisticated agent saves strictly more than her naive counterpart in the presence of period 2 income risk.

Figure 4: Percent Differences in Consumption



(a) Percent Differences in Consumption

Notes: Percent differences between sophisticated and naive hyperbolic discounting agents' average consumption $\left(\frac{\bar{c}_t^s}{\bar{c}_t^n} - 1\right) \cdot 100\%$, where $\bar{c}_t^i = \int c_t(x_t, \eta_t) d\Phi_t(x_t, \eta_t)$ for cross sectional distribution $\Phi_t(x_t, \eta_t)$.

5.2.3 Quantitative Relevance in a Multi-Period Model

We now turn to the multi-period model, where η_t obeys a Markov process and the borrowing constraint is strict ($\bar{A} = 0$). Our calibration strategy is analogous to the strategy for the deterministic multi-period model of Section 4.3 and described in Appendix C, which also contains a description of the solution method. We solve the model at an annual frequency and it turns out that non-monothonicities of policy functions are relatively mild and wash out at the aggregate level for average consumption shown in Figure 4. Differences in average consumption across the two agent types are very similar to those shown for the deterministic cake eating problem, also see Figure 3. For $\theta = 2$ the average relative difference of the sophisticated and the naive agent over age is -0.22 percent whereas this difference is positive and amounts to $+2.91$ percent for $\theta = 0.5$. As in the previously discussed three-period model, the logarithmic utility case ($\theta = 1$) yields higher consumption of the sophisticated agent with a positive average difference of 0.22 percent.

6 Concluding Remarks

Pollak (1968) shows that—irrespective of the specification of discount factors—the sophisticated agent and her naive counterpart exhibit the same savings behavior whenever their period utility function is logarithmic. We extend Pollak's analysis to the class of all iso-elastic power utility functions by showing that the sophisticated agent saves in every period a greater fraction of her wealth than her naive counterpart if and only if the resistance to inter-temporal substitution is larger than one. As a generalization of the additively time-separable life-cycle model we show that exactly the same relationship holds in a model with uninsured return risk, a portfolio choice, possibly self-imposed borrowing constraints and Epstein-Zin-Weil preferences. We also show that these results no longer hold in models with standard labor income risk and occasionally binding borrowing constraints.

We expect our findings to provide useful guidance for the interpretation of results in quantitative work where closed form solutions no longer arise but where the interpretation on the relative consumption responses of the two types of agents in θ still holds approximately. We also plan to investigate in future research the quantitative implications of the extension of our model to uninsured human capital income risk for the welfare costs of dynamic inconsistency.

Appendix

A Proofs of Theorem 1 and of Main Propositions

A.1 Proof of Theorem 1

Our proof of Theorem 1 is based on the recursive presentations of the marginal propensities to consume of the sophisticated and the naive agent. The different implications for the cases $\theta < 1$ versus $\theta > 1$ result from a simple application of Jensen's inequality to strictly concave and strictly convex functions, respectively. Because the proof of Theorem 1 will be implied by the proof of Lemma 1, we prove, at first, Lemma 1.

Proof of Lemma 1. Part (i): We show for $h \in \{0, \ldots, T-2\}$:

(i) $\theta < 1$ *implies* $m_h^n = m_h^s$ *if* $m_t^{n,h} = m_t^s$ *for all* $t \geq h+1$.

(ii) $\theta < 1$ *implies* $m_h^n < m_h^s$ *if* $m_t^{n,h} \neq m_t^s$ *for some* $t \geq h+1$.

Recall from (7) and (55) the following expressions for MPCs

$$m_h^s = \frac{1}{1 + \left(\rho_{h,h+1}\zeta_{h+1}^h\right)^{\frac{1}{\theta}}}$$

where

$$\zeta_t^h = m_t^{s^{1-\theta}} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \left(1 - m_t^s\right)^{1-\theta} \zeta_{t+1}^h \tag{46}$$

as well as

$$m_t^{n,h} = \frac{1}{1 + \left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\right)^{\frac{1}{\theta}} m_{t+1}^{n,h^{-1}}}. \tag{47}$$

Using these expressions gives us at age $t = h$

$$\begin{array}{rcl} m_h^n & \leq & m_h^s \\ & \Leftrightarrow & \\ \left(\rho_{h,h+1}\zeta_{h+1}^h\right)^{\frac{1}{\theta}} & \leq & \left(\dfrac{\rho_{h,h+1}}{\rho_{h,h}}\right)^{\frac{1}{\theta}} m_{h+1}^{n,h^{-1}} \\ & \Leftrightarrow & \\ m_{h+1}^{n,h^{\theta}} \zeta_{h+1}^h & \leq & 1. \end{array}$$

Next, we appropriately transform ζ_t^h. To this purpose, notice from (47) that

$$\frac{\rho_{h,t+1}}{\rho_{h,t}} = \left(\frac{1-m_t^{n,h}}{m_t^{n,h}}\right)^{\theta} m_{t+1}^{n,h^{\theta}}.$$

Using this in (46) we get recursively for $t = T-2, \ldots, h+1$

$$\zeta_t^h = m_t^{s^{1-\theta}} + \left(\frac{1-m_t^{n,h}}{m_t^{n,h}}\right)^{\theta} (1-m_t^s)^{1-\theta}\, m_{t+1}^{n,h^{\theta}} \zeta_{t+1}^h$$

$$\Leftrightarrow \qquad m_t^{n,h^{\theta}} \zeta_t^h = \left(\frac{m_t^{n,h}}{m_t^s}\right)^{\theta} m_t^s + \left(\frac{1-m_t^{n,h}}{1-m_t^s}\right)^{\theta} (1-m_t^s)\, m_{t+1}^{n,h^{\theta}} \zeta_{t+1}^h. \tag{48}$$

The remainder of the proof proceeds by backward induction on (48) over $t = T-1, \ldots, h+1$.

Claims: First, we claim that, for all $t \in \{h+1, \ldots, T-1\}$, $\theta < 1$ implies

$$m_t^{n,h^{\theta}} \zeta_t^h = 1 \tag{49}$$

if $m_t^{n,h} = m_t^s$ for all $t \geq h+1$.

Second, we claim that, for all $t \in \{h+1, \ldots, T-1\}$, $\theta < 1$ implies

$$m_t^{n,h^{\theta}} \zeta_t^h < 1 \tag{50}$$

if $m_t^{n,h} \neq m_t^s$ for some $t \geq h+1$.

Base Case: Recall that $m_T^n = m_T^{n,h} = m_T^s = 1$. In period $t = T-1$ we have

$$m_{T-1}^{n,h^{\theta}} \zeta_{T-1}^h = \left(\frac{m_{T-1}^{n,h}}{m_{T-1}^s}\right)^{\theta} m_{T-1}^s + \left(\frac{1-m_{T-1}^{n,h}}{1-m_{T-1}^s}\right)^{\theta} \left(1-m_{T-1}^s\right).$$

Suppose, at first, that $m_{T-1}^{n,h} = m_{T-1}^s$. Then our first claim (49) is trivially satisfied for $t = T-1$ because of

$$m_t^{n,h^{\theta}} \zeta_t^h = 1$$

irrespective of the value of θ.

Suppose now that $m_{T-1}^{n,h} \neq m_{T-1}^s$, implying

$$\frac{m_{T-1}^{n,h}}{m_{T-1}^s} \neq \frac{1-m_{T-1}^{n,h}}{1-m_{T-1}^s}.$$

By the strict version of Jensen's inequality, we obtain for $\theta < 1$

$$\begin{aligned} m_{T-1}^{n,h^\theta}\zeta_{T-1}^h &= \left(\frac{m_{T-1}^{n,h}}{m_{T-1}^s}\right)^\theta m_{T-1}^s + \left(\frac{1-m_{T-1}^{n,h}}{1-m_{T-1}^s}\right)^\theta \left(1-m_{T-1}^s\right) \\ &< \left(\left(\frac{m_{T-1}^{n,h}}{m_{T-1}^s}\right) m_{T-1}^s + \left(\frac{1-m_{T-1}^{n,h}}{1-m_{T-1}^s}\right) \left(1-m_{T-1}^s\right)\right)^\theta \\ &= 1 \end{aligned}$$

because x^θ is strictly concave for $\theta < 1$. Consequently, our second claim (50) is satisfied for $t = T-1$.

Backward Induction Step: Suppose that the first claim (49) has been proved for period $i+1$. That is, we have shown that $\theta < 1$ implies

$$m_{i+1}^{n,h^\theta}\zeta_{i+1}^h = 1 \tag{51}$$

if $m_t^{n,h} = m_t^s$ for all $t \geq i+1$. Rewrite (48) as

$$m_i^{n,h^\theta}\zeta_i^h = \underbrace{\left(\frac{m_i^{n,h}}{m_i^s}\right)^\theta m_i^s + \left(\frac{1-m_i^{n,h}}{1-m_i^s}\right)^\theta (1-m_i^s)}_{=\Lambda(m_i^{n,h}, m_i^s)} m_{i+1}^{n,h^\theta}\zeta_{i+1}^h.$$

By the same reasoning as in the base case, we have that $\theta < 1$ implies

$$\Lambda(m_i^{n,h}, m_i^s) \leq 1 \tag{52}$$

whereby this inequality is strict if and only if $m_i^{h,n} \neq m_i^s$. Since

$$x + y \leq 1 \text{ and } b \leq 1 \text{ implies } x + by \leq 1,$$

(51) together with (52) gives us the desired result that $\theta < 1$ implies

$$m_i^{n,h^\theta}\zeta_i^h = 1 \tag{53}$$

if $m_i^{h,n} = m_i^s$ whereas we have

$$m_i^{n,h^\theta}\zeta_i^h < 1$$

if $m_i^{h,n} \neq m_i^s$.

Next suppose that we have proved the second claim (50) for period $i+1$. That is, we have shown that $\theta < 1$ implies

$$m_{i+1}^{n,h^\theta} \zeta_{i+1}^h < 1$$

if $m_t^{n,h} \neq m_t^s$ for some $t \geq i+1$. Because of (52), we must have that

$$m_i^{n,h^\theta} \zeta_i^h < 1.$$

Combining both cases proves Part (i) of Lemma 1.□

Proof of Lemma 1. Part (ii): We show for $h \in \{0, \ldots, T-2\}$:

(i) $\theta > 1$ *implies* $m_h^n = m_h^s$ *if* $m_t^{n,h} = m_t^s$ *for all* $t \geq h+1$.

(ii) $\theta > 1$ *implies* $m_h^n < m_h^s$ *if* $m_t^{n,h} \neq m_t^s$ *for some* $t \geq h+1$.

The proof proceeds exactly as the proof of Part (i) of Lemma 1 whereby we prove the following two claims:

First, for all $t \in \{h+1, \ldots, T-1\}$, $\theta > 1$ implies

$$m_t^{n,h^\theta} \zeta_t^h = 1$$

if $m_t^{n,h} = m_t^s$ for all $t \geq h+1$.

Second, for all $t \in \{h+1, \ldots, T-1\}$, $\theta > 1$ implies

$$m_t^{n,h^\theta} \zeta_t^h > 1 \tag{54}$$

if $m_t^{n,h} \neq m_t^s$ for some $t \geq h+1$.

The only difference to the proof of Part (i) is the reversed strict inequality in claim (54) which follows, by the strict version of Jensen's inequality, by strict convexity of x^θ for $\theta > 1$.□□

Proof of Theorem 1. To prove Part (i), we have to show that $\theta < 1$ implies $m_h^n \leq m_h^s$. Recall from the proof of Lemma 1(i) that

$$m_t^{n,h^\theta} \zeta_t^h \leq 1 \text{ for all } t \in \{T-2, \ldots, h+1\} \text{ implies } m_h^n \leq m_h^s.$$

Moreover, the proof of Lemma 1(i) had established that $\theta < 1$ implies either $m_t^{n,h^\theta} \zeta_t^h = 1$ or $m_t^{n,h^\theta} \zeta_t^h < 1$ for all $t \in \{T-2, \ldots, h+1\}$. An analogous argument applies to Part (ii) of Theorem 1.□□

A.2 Additional Proofs and Derivations

A.2.1 Proof of Proposition 1

For $h = T$, we trivially have as optimal consumption $c_T^{*,T} = w_T$ with optimal MPC $m_T^{*,T} = 1$. For $h < T$, the optimal period h consumption $c_h^{*,h}$ from the perspective of the h-old agent is pinned down by the following FOC:

$$\left.\frac{d}{dm_h}\hat{U}_h\left(m_h, ..., m_T\right)\right|_{m_h = m_h^{*,h}} = 0$$

$$\Leftrightarrow$$

$$u'(m_h^{*,h}) = \sum_{t=h+1}^{T} \rho_{h,t} u'\left(\left(1 - m_h^{*,h}\right) m_t \prod_{j=h+1}^{t-1} (1 - m_j)\right)\left(m_t \prod_{j=h+1}^{t-1} (1 - m_j)\right),$$

which becomes for the power period utility function

$$m_h^{*,h^{-\theta}} = \left(1 - m_h^{*,h}\right)^{-\theta} \sum_{t=h+1}^{T} \rho_{h,t}\left(m_t \prod_{j=h+1}^{t-1} (1 - m_j)\right)^{1-\theta}.$$

Solving for $m_h^{*,h}$ yields the best response function

$$m_h^{*,h}\left(m_{h+1}, ..., m_T\right) = \frac{1}{1 + \left(\sum_{t=h+1}^{T} \rho_{h,t}\left(m_t \prod_{j=h+1}^{t-1}(1 - m_j)\right)^{1-\theta}\right)^{\frac{1}{\theta}}}.$$

A.2.2 Proof of Proposition 2

The proof is nested as part of the more general proof in a model with return risk provided as Proof of Proposition 4 below.

A.2.3 Proof of Proposition 3

Mathematically equivalently, the h-old naive agent's *planned* MPCs are pinned down by the following FOCs for all t such that $h \leq t < T$:

$$\rho_{h,t}(m_t^{n,h} w_t)^{-\theta} = \rho_{h,t+1}\left(m_{t+1}^{n,h} w_{t+1}\right)^{-\theta}$$

$$\Leftrightarrow$$

$$\rho_{h,t}(m_t^{n,h} w_t)^{-\theta} = \rho_{h,t+1}\left(m_{t+1}^{n,h}\left(w_t - m_t^{n,h} w_t\right)\right)^{-\theta}$$

$$\Leftrightarrow$$

$$m_t^{n,h} = \frac{1}{1 + \left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\right)^{\frac{1}{\theta}}\left(m_{t+1}^{n,h}\right)^{-1}}. \quad (55)$$

Substituting

$$m_{t+1}^{n,h} = \frac{1}{1 + \left(\frac{\rho_{h,t+2}}{\rho_{h,t+1}}\right)^{\frac{1}{\theta}} \left(m_{t+2}^{n,h}\right)^{-1}}$$

in (55) gives

$$m_t^{n,h} = \frac{1}{1 + \left(\frac{\rho_{h,t+1}}{\rho_{h,t}}\right)^{\frac{1}{\theta}} + \left(\frac{\rho_{h,t+2}}{\rho_{h,t}}\right)^{\frac{1}{\theta}} \left(m_{t+2}^{n,h}\right)^{-1}}.$$

By repeating this argument until $m_T^{n,h} = 1$, we obtain the following closed form description of planned MPCs

$$m_t^{n,h} = \begin{cases} 1 & \text{for } t = T \\ \frac{1}{1+\sum_{k=t+1}^{T}\left(\frac{\rho_{h,k}}{\rho_{h,t}}\right)^{\frac{1}{\theta}}} & \text{for } t \leq T-1. \end{cases}$$

A.2.4 Derivation of Equation (8)

The consumption growth rate of the sophisticated agent is given as

$$\frac{c_{h+1}^s}{c_h^s} = \frac{m_{h+1}^s w_{h+1}}{m_h^s w_h} = \frac{1 - m_h^s}{m_h^s} m_{h+1}^s.$$

Using the expression for m_h^s and ζ_{h+1}^h from Proposition 2 we obtain

$$\begin{aligned}
\frac{c_{h+1}^s}{c_h^s} &= \rho_{h,h+1}^{\frac{1}{\theta}} \zeta_{h+1}^{h\,\frac{1}{\theta}} \\
&= \rho_{h,h+1}^{\frac{1}{\theta}} \left(m_{h+1}^{s\,1-\theta} + \frac{\rho_{h,h+2}}{\rho_{h,h+1}} \left(1 - m_{h+1}^s\right)^{1-\theta} \zeta_{h+2}^h \right)^{\frac{1}{\theta}} m_{h+1}^s \\
&= \rho_{h,h+1}^{\frac{1}{\theta}} \left(m_{h+1}^s + \frac{\rho_{h,h+2}}{\rho_{h,h+1}} \left(1 - m_{h+1}^s\right) \left(\frac{1 - m_{h+1}^s}{m_{h+1}^s}\right)^{-\theta} \zeta_{h+2}^h \right)^{\frac{1}{\theta}} \\
&= \rho_{h,h+1}^{\frac{1}{\theta}} \left(m_{h+1}^s + \frac{\rho_{h,h+2}}{\rho_{h,h+1}} \left(1 - m_{h+1}^s\right) \left(\left(\rho_{h+1,h+2} \zeta_{h+2}^{h+1}\right)^{\frac{1}{\theta}}\right)^{-\theta} \zeta_{h+2}^h \right)^{\frac{1}{\theta}} \\
&= \rho_{h,h+1}^{\frac{1}{\theta}} \left(m_{h+1}^s + \frac{\rho_{h,h+2}}{\rho_{h,h+1}\rho_{h+1,h+2}} \frac{\zeta_{h+2}^h}{\zeta_{h+2}^{h+1}} \left(1 - m_{h+1}^s\right) \right)^{\frac{1}{\theta}}.
\end{aligned}$$

Noting that $u_c(c) = c^{-\theta}$ then gives (8).

A.2.5 Proof of Proposition 4

Sophisticated Agent. Our proof is by backward induction.

Claims: The value function of the sophisticated agent in any period $t \geq h$ is given by

$$U_t^h(w_t) = \frac{1}{1-\theta}\zeta_t^h w_t^{1-\theta} \tag{56}$$

with associated policy function

$$c_h^s = m_h^s w_h. \tag{57}$$

Base case: In period T we have $c_T^s = w_T$ and thus $U_T^h = \frac{1}{1-\theta} w_T^{1-\theta}$ and $m_T^s = 1$.

Backward Induction Steps: Suppose the claims (56) and (57) have been shown for all periods $h+1, \ldots, T$. Then iterate backward for all $t = T-1, \ldots, h+1$ using (56) in (28) to get, also using resource constraint (32),

$$\begin{aligned}
U_t^h &= u(c_t) + \frac{\rho_{h,t+1}}{\rho_{h,t}} \frac{1}{1-\theta} \left(\mathbb{E}\left[\left((1-\theta)\, U_{t+1}^h \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \\
&= \frac{1}{1-\theta} \left((c_t^s)^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \zeta_{t+1}^h \left(\mathbb{E}\left[\left(w_{t+1}^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \right) \\
&= \frac{1}{1-\theta} \left((m_t^s)^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \left(1 - m_t^s\right)^{1-\theta} \zeta_{t+1}^h \Theta\left(\hat{\alpha}_t, R^f, R_{t+1}, \pi\right) \right) w_t^{1-\theta} \\
&= \frac{1}{1-\theta} \zeta_t^h w_t^{1-\theta},
\end{aligned} \tag{58}$$

which defines (36) and establishes the backward recursion of ζ_t^h in (35).

Next, in period h use (56) in (28) to get

$$U_h^h = \frac{1}{1-\theta} \max_{c_h^s, w_{h+1}, \hat{\alpha}_h^s} \left\{ (c_h^s)^{1-\theta} + \rho_{h,h+1} \zeta_{h+1}^h \left(\mathbb{E}\left[\left(w_{h+1}^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \right\}. \tag{59}$$

Use the resource constraint (32) in the above to obtain, by the separation between the optimal consumption and the optimal portfolio choice,

$$U_h^h = \frac{1}{1-\theta} \max_{c_h^s} \left\{ (c_h^s)^{1-\theta} + \rho_{h,h+1} \left(w_h - c_h^s \right)^{1-\theta} \right\} \zeta_{h+1}^h \underbrace{\max_{\hat{\alpha}_h} \left\{ \left(\mathbb{E}\left[\left(R_{h+1}^p (\hat{\alpha}_h)^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right] \right)^{\frac{1-\theta}{1-\sigma}} \right\}}_{=\Theta\left(\hat{\alpha}_h, \zeta_{h+1}^h, R^f, R_{h+1}, \pi\right)}$$

with first-order condition for c_h^s

$$\left(c_h^s\right)^{-\theta} - \rho_{h,h+1} \left(w_h - c_h \right)^{-\theta} \zeta_{h+1}^h \Theta\left(\hat{\alpha}_h, R^f, R_{h+1}, \pi\right) = 0,$$

where $\hat{\alpha}_h^*$ is the optimal portfolio share further characterized below. We thus get

$$c_h^s = m_h^s w_h$$

where

$$m_h^s = \frac{1}{1 + \left[\rho_{h,h+1}\zeta_{h+1}^h \Theta\left(\hat{\alpha}_h, R^f, R_{h+1}, \pi\right)\right]^{\frac{1}{\theta}}}.$$

which is (59) and proves the claims.

Naive Agent. For the naive agent, we essentially follow the same steps with the following modifications:

- The maximization problem in (59) is solved for all $t = h, \dots, T-1$, thus

$$U_t^{n,h} = \frac{1}{1-\theta} \max_{c_t^{n,h}, w_{t+1}, \hat{\alpha}_t^{n,h}} \left\{ \left(c_t^{n,h}\right)^{1-\theta} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \zeta_{t+1}^h \left(\mathbb{E}\left[\left(w_{t+1}^{1-\theta}\right)^{\frac{1-\sigma}{1-\theta}}\right]\right)^{\frac{1-\theta}{1-\sigma}} \right\},$$

  which, using the resource constraint and the separation between the optimal consumption and the portfolio choice, gives

$$m_t^{n,h} = \frac{1}{1 + \left[\frac{\rho_{h,t+1}}{\rho_{h,t}} \zeta_{t+1}^h \Theta\left(\hat{\alpha}_t, R^f, R_{t+1}, \pi\right)\right]^{\frac{1}{\theta}}}. \tag{60}$$

- Using the solution back in the value function as in (58) gives

$$\begin{aligned} U_t^{n,h} &= \frac{1}{1-\theta}\left(m_t^{n,h^{1-\theta}} + \frac{\rho_{h,t+1}}{\rho_{h,t}} \zeta_{t+1}^{n,h} \left(1 - m_t^{n,h}\right)^{1-\theta} \Theta\left(\hat{\alpha}_t^{n,h}, R^f, R_{t+1}, \pi\right)\right) w_t^{1-\theta} \\ &= \frac{1}{1-\theta}\left(m_t^{n,h^{1-\theta}} + \left(1 - m_t^{n,h}\right)^{1-\theta} \left(\frac{1 - m_t^{n,h}}{m_t^{n,h}}\right)^{\theta}\right) w_t^{1-\theta} \\ &= \frac{1}{1-\theta} m_t^{n,h^{-\theta}} w_t^{1-\theta}. \end{aligned}$$

- We thus find $\zeta_t^h = m_t^{h^{-\theta}}$. Using this in (60) we finally obtain

$$m_t^h = \frac{1}{1 + \left(\frac{\rho_{h,t+1}}{\rho_{h,t}} \Theta\left(\hat{\alpha}_t^{n,h}, R^f, R_{t+1}, \pi\right)\right)^{\frac{1}{\theta}} \left(m_{t+1}^{n,h}\right)^{-1}}.$$

Optimal Portfolio Choice. Since $\Theta\left(\hat{\alpha}_t, R^f, R_{t+1}, \pi\right)$ is the same for both agents we obtain $\hat{\alpha}_t^s = \hat{\alpha}_t^{n,h} = \hat{\alpha}_t$, where from the first-order condition of the optimal portfolio allocation problem $\hat{\alpha}_t^*$ is the solution to

$$\mathbb{E}\left[R_{t+1}^p(\hat{\alpha}_t)^{-\sigma}\right] = \int R_{t+1}^p(\hat{\alpha}_t)^{-\sigma} d\pi = 0$$

and thus the optimal portfolio allocation problem at t is a static decision problem, which is parameterized by risk aversion σ.

B The Three-period Model: First-Order and Comparative Statics Analysis

B.1 First-order Analysis

The three-period model lends itself to a first-order analysis because all deviations from the dynamically consistent benchmark case can be described through one parameter, denoted ϵ, only. The three-period model is dynamically inconsistent (at age 0) if and only if

$$\text{either } \frac{\rho_{0,2}}{\rho_{0,1}} > \rho_{1,2} \text{ or } \frac{\rho_{0,2}}{\rho_{0,1}} < \rho_{1,2}, \tag{61}$$

which is the case if and only if either one of the following two cases holds:

- **Case 1:**

$$\frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}} = 1 + \epsilon \text{ for some } \epsilon > 0, \tag{62}$$

- **Case 2:**

$$\frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}} = 1 - \epsilon \text{ for some } \epsilon \in (0, 1).$$

Let us start with Case 1. Pick arbitrary values of $\rho_{0,1}$, $\rho_{1,2}$, and $\rho_{0,2}$ satisfying $\frac{\rho_{0,2}}{\rho_{0,1}} > \rho_{1,2}$, which pins down a unique value of $\epsilon > 0$ in (62). Fix the discount factors $\rho_{0,1}, \rho_{1,2}$ and rewrite both MPCs as the following functions in $\varepsilon \in [0, \epsilon]$:

$$m_0^n(\varepsilon) = \frac{1}{1 + \rho_{0,1}^{\frac{1}{\theta}} \left(1 + (1+\varepsilon)^{\frac{1}{\theta}} \rho_{1,2}^{\frac{1}{\theta}}\right)},$$

$$m_0^s(\varepsilon) = \frac{1}{1 + \rho_{0,1}^{\frac{1}{\theta}} \left(\left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}\right)^{1-\theta} + \rho_{1,2}^{\frac{1}{\theta}} \left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}\right)^{1-\theta} (1+\varepsilon)\right)^{\frac{1}{\theta}}}.$$

Note that

$$m_0^n(0) = m_0^s(0) = \frac{1}{1 + \rho_{0,1}^{\frac{1}{\theta}} \left(1 + \rho_{1,2}^{\frac{1}{\theta}}\right)}$$

as well as

$$m_0^n(\epsilon) = m_0^n = \frac{1}{1+\rho_{0,1}^{\frac{1}{\theta}}\left(1+\left(\frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}}\right)^{\frac{1}{\theta}}\rho_{1,2}^{\frac{1}{\theta}}\right)},$$

$$m_0^s(\epsilon) = m_0^s = \frac{1}{1+\rho_{0,1}^{\frac{1}{\theta}}\left(\left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}\right)^{1-\theta}+\rho_{1,2}^{\frac{1}{\theta}}\left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}\right)^{1-\theta}\left(\frac{\rho_{0,2}}{\rho_{0,1}\rho_{1,2}}\right)\right)^{\frac{1}{\theta}}}.$$

That is, for fixed $\rho_{0,1}, \rho_{1,2}$ we describe both MPCs as functions in $\varepsilon \in [0, \epsilon]$ such that $\varepsilon = 0$ stands for the dynamically consistent benchmark case—disregarding the true value of $\rho_{0,2}$—whereas $\varepsilon = \epsilon$ yields the true—but dynamically inconsistent—value $\rho_{0,2} = (1+\epsilon)\rho_{0,1}\rho_{1,2}$. Our analytical strategy is now very simple: Since we have for the differentiable function

$$f(\varepsilon) \equiv m_0^s(\varepsilon) - m_0^n(\varepsilon)$$

that $f(0) = 0$, any parameter conditions that imply (i) a non-negative slope $\frac{d}{d\varepsilon}f(\varepsilon) \geq 0$ for $\varepsilon = 0$ as well as (ii) a strictly positive slope $\frac{d}{d\varepsilon}f(\varepsilon) > 0$ for all $\varepsilon \in (0, \epsilon]$ will ensure that

$$f(\epsilon) > 0 \Leftrightarrow m_0^n < m_0^s.$$

For analytical convenience define $q_0^i(\varepsilon) \equiv \frac{1}{\rho_{0,1}^{\frac{1}{\theta}}}\frac{1-m_0^i(\varepsilon)}{m_0^i(\varepsilon)}$ for $i \in \{n, s\}$ so that

$$q_0^n(\varepsilon) \equiv 1 + (1+\varepsilon)^{\frac{1}{\theta}}\rho_{1,2}^{\frac{1}{\theta}} \qquad \text{and} \qquad q_0^s(\varepsilon) \equiv \left(\left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}\right)^{1-\theta} + \rho_{1,2}^{\frac{1}{\theta}}\left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}\right)^{1-\theta}(1+\varepsilon)\right)^{\frac{1}{\theta}}$$

and observe that $\frac{d}{d\varepsilon}q_0^s(\varepsilon) \leq \frac{d}{d\varepsilon}q_0^n(\varepsilon)$ is equivalent to $\frac{d}{d\varepsilon}f(\varepsilon) \geq 0$. Taking the first-order derivatives yields

$$\frac{d}{d\varepsilon}q_0^n(\varepsilon) = \frac{1}{\theta}\rho_{1,2}^{\frac{1}{\theta}}(1+\varepsilon)^{\frac{1}{\theta}-1} > 0 \qquad \text{and} \qquad \frac{d}{d\varepsilon}q_0^s(\varepsilon) = \frac{1}{\theta}\rho_{1,2}^{\frac{1}{\theta}}\left(1+(1-m_1)\varepsilon\right)^{\frac{1}{\theta}-1} > 0,$$

where $1 - m_1 = \frac{\rho_{1,2}^{\frac{1}{\theta}}}{1+\rho_{1,2}^{\frac{1}{\theta}}} \in (0,1)$, implying (i) $\frac{d}{d\varepsilon}f(0) = 0$ as well as (ii) for all $\varepsilon \in (0, \epsilon]$

$$\frac{d}{d\varepsilon}f(\varepsilon) > 0 \qquad \Leftrightarrow \qquad (1+(1-m_1)\varepsilon)^{1-\theta} < (1+\varepsilon)^{1-\theta} \qquad \Leftrightarrow \qquad \theta < 1.$$

For Case 1 we thus obtain that

$$\theta < (>)\, 1 \text{ implies } m_0^n < (>)\, m_0^s \text{ whenever } \frac{\rho_{0,2}}{\rho_{0,1}} > \rho_{1,2}.$$

Turn now to Case 2 for which both MPCs become the following functions in $\varepsilon \in [0, \epsilon]$:

$$m_0^n(\varepsilon) = \frac{1}{1 + \rho_{0,1}^{\frac{1}{\theta}} \left(1 + (1-\varepsilon)^{\frac{1}{\theta}} \rho_{1,2}^{\frac{1}{\theta}}\right)},$$
$$m_0^s(\varepsilon) = \frac{1}{1 + \rho_{0,1}^{\frac{1}{\theta}} \left(\left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}\right)^{1-\theta} + \rho_{1,2}^{\frac{1}{\theta}} \left(\frac{1}{1+\rho_{1,2}^{\frac{1}{\theta}}}\right)^{1-\theta} (1-\varepsilon)\right)^{\frac{1}{\theta}}}.$$

By an analogous argument as for Case 1, we have

$$\frac{d}{d\varepsilon} q_0^s(\varepsilon) < \frac{d}{d\varepsilon} q_0^n(\varepsilon) \text{ for all } \varepsilon \in (0,1) \qquad \Rightarrow \qquad m_0^n(\varepsilon) < m_0^s(\varepsilon),$$

where

$$\frac{d}{d\varepsilon} q_0^n(\varepsilon) = -\frac{1}{\theta} \rho_{1,2}^{\frac{1}{\theta}} (1-\varepsilon)^{\frac{1}{\theta}-1} < 0 \qquad \text{and} \qquad \frac{d}{d\varepsilon} q_0^s(\varepsilon) = -\frac{1}{\theta} \rho_{1,2}^{\frac{1}{\theta}} \left(1 - (1-m_1)\varepsilon\right)^{\frac{1}{\theta}-1} < 0,$$

implying

$$\frac{d}{d\varepsilon} q_0^s(\varepsilon) < \frac{d}{d\varepsilon} q_0^n(\varepsilon) \qquad \Leftrightarrow \qquad (1-(1-m_1)\varepsilon)^{1-\theta} > (1-\varepsilon)^{1-\theta} \qquad \Leftrightarrow \qquad \theta < 1.$$

This gives us the statement

$$\theta < (>)\, 1 \text{ implies } m_0^n < (>)\, m_0^s \text{ whenever } \frac{\rho_{0,2}}{\rho_{0,1}} < \rho_{1,2}.$$

Combining Case 1 with Case 2 establishes the desired relationship

$$\theta < (>)\, 1 \text{ implies } m_0^n < (>)\, m_0^s \text{ whenever } \frac{\rho_{0,2}}{\rho_{0,1}} \neq \rho_{1,2}. \tag{63}$$

B.2 The Comparative Statics Analysis in Salanié and Treich (2006)

Restricted to the three-period model, Salanié and Treich (2006) (henceforth ST) already derive our finding for the deterministic cake-eating problem. In their model the life-cycle utility functions of the 0-old and the 1-old agent are respectively given as

$$\begin{aligned} U_0(c_0, c_1) &= u_0(c_0) + u_1(c_1) + \mu u_2(w_0 - c_0 - c_1) \\ U_1(c_0, c_1) &= u_1(c_1) + \lambda u_2(w_0 - c_0 - c_1) \end{aligned}$$

for arbitrary thrice-differentiable u_j. ST define 'lack of self-control' as the inequality $\lambda \neq \mu$. Adopted to our three-period framework with power period utility u, the maximization of the following ST life-cycle utility functions

$$\begin{aligned} U_0(c_0, c_1) &= \frac{1}{\rho_{0,1}} u(c_0) + \frac{\rho_{0,1}}{\rho_{0,1}} u(c_1) + \frac{\rho_{0,2}}{\rho_{0,1}} u(w_0 - c_0 - c_1), \\ U_1(c_0, c_1) &= u(c_1) + \rho_{1,2} u(w_0 - c_0 - c_1) \end{aligned}$$

becomes equivalent to the utility maximization of our three-period model. For the three-period model the ST definition of 'lack of self-control' is thus equivalent to our definition of dynamic inconsistency (11) because of

$$\lambda \neq \mu \Leftrightarrow \rho_{1,2} \neq \frac{\rho_{0,2}}{\rho_{0,1}}.$$

Rewrite the period 0 sophisticated agent's utility $U_0(c_0, c_1^*(c_0))$ as a function in c_0, λ, denoted $U_0(c_0, \lambda)$. By a single-crossing argument from comparative statics analysis (cf. Theorem 4 in Milgrom and Shannon 1994), the maximizers

$$\{c_0^s\} = \arg\max_{c_0} U_0(c_0, \lambda)$$

increase (resp. decreases) in λ iff the cross-derivative satisfies

$$\frac{\partial}{\partial c_0} \frac{\partial U_0(c_0, \lambda)}{\partial \lambda} \geq 0 \text{ (resp. } \leq 0).$$

Because of $c_0^s = c_0^n$ for $\lambda = \mu$, we thus have that $c_0^s < c_0^n$ if either

$$\lambda < \mu \text{ and } \frac{\partial}{\partial c_0} \frac{\partial U_0(c_0, \lambda)}{\partial \lambda} > 0$$

or

$$\lambda > \mu \text{ and } \frac{\partial}{\partial c_0} \frac{\partial U_0(c_0, \lambda)}{\partial \lambda} < 0$$

(cf. Figure 1 in ST). Although this comparative statics argument is closely related to our own first-order analysis of the three-period model[18], it comes with the advantage that one does not require an explicit analytical solution for the maximizers c_0^s and c_0^n which we use (for the special case of power utility) in our first-order analysis. As a consequence, ST can obtain the following powerful result for general u_j:

[18]Whereas, e.g., our Case 1 first-order analysis keeps $\rho_{0,1}, \rho_{1,2}$ fixed and considers deviations $(1+\varepsilon)\rho_{0,1}\rho_{1,2}$, $\varepsilon \in [0, \epsilon]$, from the dynamic consistency benchmark $\varepsilon = 0$ towards the true value $\rho_{0,2} = (1+\epsilon)\rho_{0,1}\rho_{1,2}$, ST keep $\mu = \frac{\rho_{0,2}}{\rho_{0,1}}$ fixed and consider deviations in λ from the dynamic consistency benchmark $\lambda = \mu$ towards the true value $\lambda = \rho_{1,2}$. By the continuity of the problem, it does not matter from which initial values of discount factors the true values $\rho_{0,1}, \rho_{1,2}, \rho_{0,2}$ are approached as long as the desired strict inequalities for m_0^n (i.e., c_0^n) versus m_0^s (i.e., c_0^s) can be established.

Salanié and Treich (Proposition 1, 2006). *Any 'lack of self-control' $\lambda \neq \mu$, i.e., any dynamic inconsistency in the three-period model, reduces the savings of the sophisticated* 0*-old agent compared to her naive counterpart if and only if, for all* $c > 0$,

$$-\frac{u_j'''(c)}{u_j''(c)} \geq -2\frac{u_j''(c)}{u_j'(c)} \text{ for } j = 2, 3. \tag{64}$$

ST proceed by observing that (i) (64) holds with equality if and only if u_j is the logarithmic function and (ii) that (64) holds—within our class of power-period utility functions—with strict inequality if and only if $\theta < 1$. In other words, the monotone comparative statics analysis in ST arrives through an alternative proof at our main finding (63) for the three-period baseline model.

As a drawback, however, any monotone comparative statics analysis (like our first-order analysis) becomes intractable for more than three periods. The problem is that the number of inequalities—and the corresponding deviation parameters—that are needed to cover all possibilities of dynamic inconsistency explode once we move beyond three periods. To see this for the four-period model, observe that instead of the two inequalities (61) we must now keep track of all the combinations of inequalities

$$\frac{\rho_{h,t+1}}{\rho_{h,t}} < \rho_{t,t+1} \text{ or } \frac{\rho_{h,t+1}}{\rho_{h,t}} > \rho_{t,t+1} \text{ with } h \in \{0,1\} \text{ and } t > h,\, t \leq 2.$$

Whereas dynamic inconsistency can thus no longer be tackled for $T \geq 3$ through first order or/and single-crossing analysis, the recursive structure analysis of the present paper provides an elegant solution to the problem for arbitrary $T < \infty$.

An alternative to our recursive proof for $T \geq 3$ is a proof based on a variational argument, which is presented in the earlier working paper version of this paper (Groneck, Ludwig, and Zimper 2021), by which we can establish under dynamic inconsistency a weak inequality at all ages, i.e. $\theta > 1$ implies $m_h^s \leq m_h^n$ for all $h = 0, \ldots, T-2$ and vice versa for $\theta < 1$. The main advantages of our recursive proof are, first, that we can show that these inequalities are strict and, second, that the main proof idea straightforwardly extends to models with return risk.

C Calibration of Quantitative Models and Solution Method

C.1 Calibration

Agents become economically active in period 0 (corresponding to biological age 30) with initial assets of $a_0 = 3.4$, cf. Groneck, Ludwig, and Zimper (2016) for data sources, and live with certainty up to period T (corresponding to biological age 80). During the working period, they receive stochastic labor income normalized to a mean of 1, thus $\mathbb{E}\eta_h = 1$ until retirement at age $h_r = 35$. Calibration of the Markov shock process $\left\{\eta_h, \pi(\eta_{h+1} \mid \eta_h), \Pi(\eta_0)\right\}$ is based on the estimates of Busch and Ludwig (2023) for the PSID for the moments of an $AR(1)$ process with an autocorrelation coefficient $\rho = 0.9683$ and a variance of the income shock of $\sigma_\epsilon^2 = 0.1165$.[19] We discretize the $AR(1)$ process via Tauchen's method assuming $n = 11$ states.[20] The replacement rate of the pension system is set to $\rho = 0.5$. To compare different models varying $\theta_i \in \{0.5, 1, 2\}$ we calibrate the long-run discount factor δ so that the aggregate asset holdings in the model economy match smoothed SCF data on assets taken from Groneck, Ludwig, and Zimper (2016). Assets in the model are evaluated by assuming equal shares of naive and sophisticated hyperbolic time discounters with long-run discount factor δ and short-run discount factor β whereby we hold constant the difference between the short and the long-run discount rates Δ as calibrated by Angeletos et al. (2001), i.e., $\Delta = 1/0.7 - 1/0.95i7 = 0.383$ leaving δ as the only free parameter. Matching asset profiles yields the parameters summarized in Table 2 for each model variant. Note that in our model with a zero interest rate a long-run discount factor above one is not surprising.

C.2 Numerical Solution of Stochastic Model Variant

We reformulate the dynamic problem in terms of cash-on-hand $x_h = a_h + y_h$. Denote by $c_h(x_h, \eta_h)$ the age h consumption policy function at x_h and current period income shock η_h, and by $m^s(x_{h+1}, \eta_{h+1}) = \frac{\partial c^s_{h+1}(x_{h+1}, \eta_{h+1})}{\partial x_{h+1}}$ the (now generally non-linear) marginal propensity to consume out of cash-on-

[19] Busch and Ludwig (2023) estimate a higher-order income risk process where higher order terms are orthogonal. Also, they distinguish between aggregate state dependent transitory and persistent income shocks. Our calibration is a standard approximation to this process evaluated at the stationary invariant distribution of the aggregate state.

[20] A high number of discretization nodes is chosen to smooth out any non-monotonicities of policy functions.

Table 2: Calibrated Discount Factors

	Stochastic		Deterministic	
θ	δ	β	δ	β
0.5	1.0290	0.7377	1.0329	0.7397
1.0	1.0360	0.7413	1.0469	0.7469
2.0	1.0445	0.7457	1.0772	0.7623

Notes: Calibration parameters of the short and long-term discount factors β and δ in the stochastic and deterministic variants of the QHD model for different values of θ.

hand. Since we assume QHD preferences, solution of the first-order condition gives rise to the familiar generalized Euler equation (Harris and Laibson 2001)

$$u_c(c_h^s(x_h, \eta_h) \geq \beta\delta\mathbb{E}_{\eta_{h+1}|\eta_h}\left[\left(m_{h+1}^s(x_{h+1}, \eta_{h+1}) + \frac{1}{\beta}\left(1 - m_{h+1}^s(x_{h+1}, \eta_{h+1})\right)\right) u_c(c_{h+1}^s(x_{h+1}, \eta_{h+1}))\right].$$

We solve the model by iterating on the Euler equation using Carroll (2006)'s endogenous grid method and evaluate the MPCs numerically on the consumption policy functions by the finite difference method.

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